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Engineering Mathematics - 1

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Assessment
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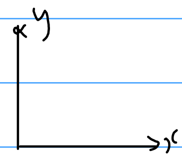
Discussion

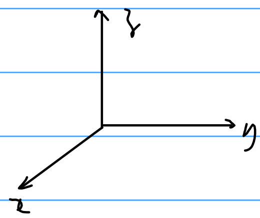
Facilitate learning → Ask questions

Differential Calculus - 1

* Derivatives. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

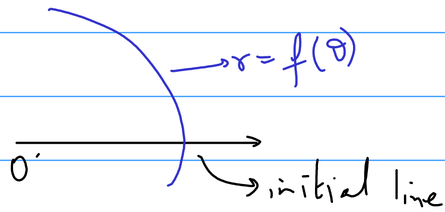




* Polar curves (also derivatives)

→ polar coordinates ✓

Polar curves → curves in terms of r and θ



→ Differential Calculus - 2

Room you are sitting 3-D → Temperature T

T = function of (x, y, z, t)

Derivative of T w.r.to x or y or z or t

How to represent surfaces [We know $y = f(x)$

$z = f(x, y)$

Maxima and Minima $\longrightarrow$ [$y = f(x) \Rightarrow y' = 0$ $\to$ y''

y'': >0 → Min; $=0$; <0 → Max

$z = f(x, y) \longrightarrow$ Max & Minima

Multiple Integrals

$\int y\,dx$ or $\int f(x)\,dx$

Ordinary differential equations of 1^{st} order

Ordinary differential equations of higher order

Syllabus copy $\longrightarrow$ Will be Mailed to you.
or
Google Classroom

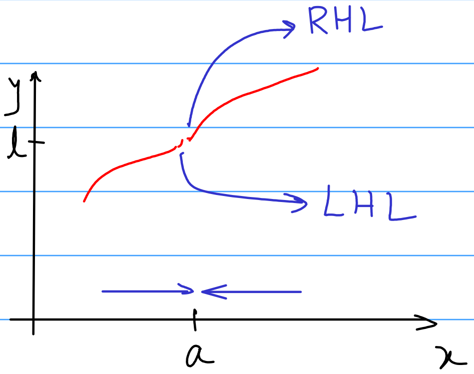
Revi

Revision

$y = f(x)$ iff $y: A \longrightarrow B$ where every element of A is associated with unique element in B

$A, B \subseteq \mathbb{R}$ then we call it real valued function

$\lim_{x \to a} f(x) = l$ $\quad |f(x) - l| < \varepsilon$ if $|x - a| < \delta$



RHL = LHL = l then limit exists

Continuity $\quad \lim_{x \to a} f(x) = f(a)$

If $f(a)$ not defined $\longrightarrow$ removable discontinuity

Differentiability $\quad \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{df}{dx}$

$y = f(x)$ $\quad \left.\frac{dy}{dx}\right|_{x = x_0}$ = slope of tangent at x_0

tangent line : $y = mx + c$ $\quad m = \left.\frac{dy}{dx}\right|_{x = x_0}$

bending of the curve, $\frac{d^2y}{dx^2}$

List of Derivatives

y	$\frac{dy}{dx}$
e^x	e^x
x^n	nx^{n-1}
a^x	$a^x \log(a)$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\operatorname{cosec}(x)$	$-\operatorname{cosec}(x)\cot(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\cot(x)$	$-\operatorname{cosec}^2(x)$

y	$\frac{dy}{dx}$
$\log(x) = \ln(x)$	$\frac{1}{x}$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$

$\log N = \log_e N$

$\log_a N$

Hyperbolic functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \operatorname{cosech}(x) = \frac{1}{\sinh(x)}$$

Similarly $\quad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{1}{\tanh(x)}$

$\underbrace{\sinh}_{\text{Name}}(x) \neq \sin(\underbrace{h}_{\text{constant}}x)$

HW: Derivatives of hyperbolic functions

$$\frac{d}{dx}\{\sinh(x)\} = \frac{d}{dx}\left\{\frac{e^x - e^{-x}}{2}\right\} = \frac{e^x - e^{-x}(-1)}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{d}{dx}\{\cosh(x)\} = \frac{d}{dx}\left\{\frac{e^x + e^{-x}}{2}\right\} = \frac{e^x + e^{-x}(-1)}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x)$$

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
$\sinh(x)$	$\cosh(x)$	$\operatorname{cosech}(x)$	$-\operatorname{cosech}(x)\coth(x)$
$\cosh(x)$	$\sinh(x)$	$\operatorname{sech}(x)$	$-\operatorname{sech}(x)\tanh(x)$
$\tanh(x)$	$\operatorname{sech}^2(x)$	$\coth(x)$	$-\operatorname{cosech}^2(x)$

Linearity Property

$$\frac{d}{dx}(au+bv) = a\frac{du}{dx} + b\frac{dv}{dx}$$

Product rule

$$\frac{d}{dx}(uv) = \left(\frac{du}{dx}\right)v + u\frac{dv}{dx}$$

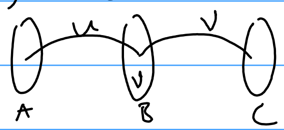
$$\frac{d}{dx}(uvw) = \frac{d}{dx}\left\{(uv)w\right\} = \frac{d(uv)}{dx}w + (uv)\frac{dw}{dx}$$

$$= \left(\frac{du}{dx}\right)vw + u\frac{dv}{dx}w + (uv)\frac{dw}{dx}$$

Chain rule

$u = u(v) \quad v = v(t)$. then $\dfrac{du}{dt} = \left(\dfrac{du}{dv}\right)\left(\dfrac{dv}{dt}\right) = D_v(u)\,D_t(v)$

$$u \longrightarrow v \longrightarrow t$$



Quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}\left\{u\left(\frac{1}{v}\right)\right\} = \frac{du}{dx}\left(\frac{1}{v}\right) + u\frac{d}{dx}\left(\frac{1}{v}\right)$$

$$= \frac{du}{dx}\frac{1}{v} + u(-1)v^{-1-1}\frac{dv}{dx}$$

$$= \frac{du}{dx}\frac{1}{v} - \frac{u}{v^2}\frac{dv}{dx}$$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

$g = \dfrac{1}{v} \qquad v = v(x)$

$\dfrac{dg}{dx} = \dfrac{dg}{dv}\dfrac{dv}{dx}$

Eg:- $y = \underbrace{x^3}_{u}\underbrace{\sin(2x)}_{v}$, find $\dfrac{d^2y}{dx^2} = y_2$

$$y_1 = \frac{dy}{dx} = (3x^2)\sin(2x) + x^3\{\cos(2x).2\}$$

$(uv)' = u'v + uv'$

$\dfrac{dx^n}{dx}$

$$y_1 = 3\underbrace{x^2}_{u}\underbrace{\sin(2x)}_{v} + 2\underbrace{x^3}_{u}\underbrace{\cos(2x)}_{v}$$

$$y_2 = \frac{d}{dx}(y_1) = 3\left[(2x)\sin(2x) + x^2\,2\cos(2x)\right] + 2\left[(3x^2)\cos(2x) + x^3\{-\sin(2x)2\}\right]$$

$$y_2 = (6x - 4x^3)\sin(2x) + 12x^2\cos(2x)$$

If $y = \dfrac{\tan(e^{2x})}{1+x}$ then find $\dfrac{dy}{dx}$

$y = \dfrac{u}{v}$ either apply product rule or quotient rule

Ans:- $$\frac{dy}{dx} = \underline{\frac{d}{dx}\left\{\tan(e^{2x})\right\}}\left(\frac{1}{1+x}\right) + \tan(e^{2x})\frac{d}{dx}\left(\frac{1}{1+x}\right)$$

↳ apply chain rule

$$= \sec^2(e^{2x})\left\{e^{2x}\cdot 2\right\}\frac{1}{1+x} + \tan(e^{2x})\left\{\frac{-1}{(1+x)^2}\cdot(1)\right\}$$

$$= 2\frac{e^{2x}\sec^2(e^{2x})}{1+x} - \frac{\tan(e^{2x})}{(1+x)^2}$$

$$\frac{d}{dx}\left\{\frac{\sin^{-1}(x)}{x\log x}\right\} = \frac{1}{\sqrt{1-x^2}}\,\frac{1}{x\log x} - \frac{\sin^{-1}(x)\left\{1+\log(x)\right\}}{x^2(\log x)^2}$$

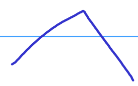
Mean value theorems

* Rolle's theorem

If $f(x)$ is defined in $[a,b]$ such that

i) $f(x)$ is continuous in $[a,b]$

No break in curve



ii) $f(x)$ is differentiable in (a,b)

smooth curve

iii) $f(a) = f(b)$

then there exist at least one point $c \in (a,b)$ where $f'(c) = 0$

Slope of tangent to the curve at $c = 0$

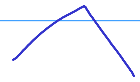
ie tangent to the curve at $c \parallel x$-axis

Lagrange's MVT

If $f(x)$ is defined in $[a,b]$ such that

i) $f(x)$ is continuous in $[a,b]$

No break in curve

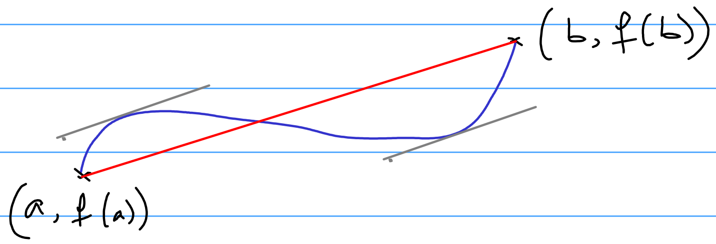


ii) $f(x)$ is differentiable in (a,b)

smooth curve

then there exist at least one point $c \in (a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



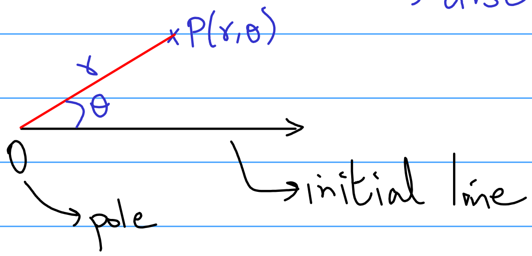
Cauchy's MVT

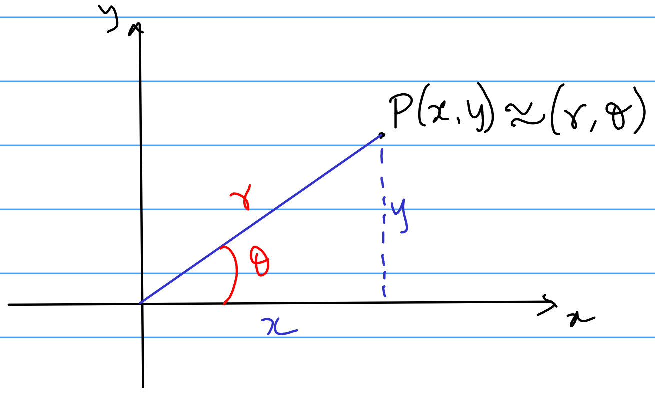
$$h = \frac{f(x)}{g(x)}$$

Polar coordinates (r, θ)

$\theta \rightarrow$ angle made by radial vector with the initial line

$r \rightarrow$ distance from a fixed point, pole





$$x = r\cos\theta$$
$$y = r\sin\theta$$

OR

$$x^2 + y^2 = r^2$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$y = f(x) \longrightarrow$ Cartesian curve $\{$ or $f(x,y) = c$

$r = f(\theta)$ or $f(r,\theta) = c \longrightarrow$ Polar curve

Circle $\rightarrow$ $x^2 + y^2 = k^2$ $\xrightarrow{\text{polar curve}}$ $\left[(r\cos\theta)^2 + (r\sin\theta)^2 = k^2\right] \Longrightarrow r = k$

$\boxed{k > 0}$

$r =$ constant represents a circle in polar coordinates with centre at pole

$$y^2 = 4ax \xrightarrow{\text{polar form}} \left[(r\sin\theta)^2 = 4ar\cos\theta\right] \Longrightarrow r = 4a\cot(\theta)\,\text{cosec}(\theta)$$

$y = f(x)$ or $f(x,y) = c \longrightarrow$ tangent line $\Rightarrow$ Slope of tangent

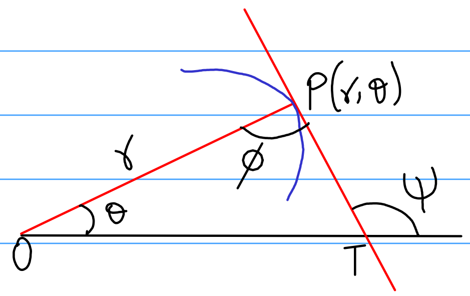
Slope of the tangent, $\tan\psi = \dfrac{dy}{dx}$

$\psi \to$ psi

$\phi \to$ phi

$\pi \to$ pi

Angle between radius vector (any line passing through the pole) and tangent



Consider a polar curve

$$r = f(\theta) \longrightarrow (1)$$

PT is tangent at $P(r,\theta)$ making an angle ψ with the initial line

From the fig $\boxed{\psi = \theta + \phi}$

Take tan on both sides

$$\tan\psi = \tan(\theta+\phi)$$

$$\frac{dy}{dx} = \frac{\tan\theta + \tan\phi}{1 - \tan(\theta)\tan(\phi)} \longrightarrow (2)$$

$x = r\cos\theta$ and $y = r\sin\theta$

$\Longrightarrow x = f(\theta)\cos\theta \quad y = f(\theta)\sin(\theta)$ {∵ using (1)

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \longrightarrow (3)$$

$$\left.\begin{aligned}\frac{dy}{d\theta} &= f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)\\ \frac{dx}{d\theta} &= f'(\theta)\cos(\theta) + f(\theta)\{-\sin(\theta)\} = f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)\end{aligned}\right\} \to (4)$$

(4) in (3) $\Longrightarrow \dfrac{dy}{dx} = \dfrac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$

÷ Numerator and denominator by $f'(\theta)\cos(\theta)$

$$\frac{dy}{dx} = \frac{\tan(\theta) + f(\theta)/f'(\theta)}{1 - \tan(\theta)\left\{f(\theta)/f'(\theta)\right\}} \longrightarrow (5)$$

$$\tan\phi = \frac{f(\theta)}{f'(\theta)} = \frac{r}{\left(dr/d\theta\right)} = r\frac{d\theta}{dr}$$

Alternately $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

Note : $\frac{d}{d\theta}\{\log(r)\} = \frac{1}{r}\frac{dr}{d\theta}$

Q) Find the slope of the tangent at any point on the curve $r = a(1+\cos\theta)$

Ans:- slope of the tangent, $\tan\psi = \tan(\theta+\phi)$ ↳ we need this

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{d}{d\theta}\{\log r\}$$

$$r = a\{1+\cos(\theta)\} \Longrightarrow \log(r) = \log\left[a\{1+\cos(\theta)\}\right]$$

$$\log(AB) =$$

$$\log r = \log a + \log\{1+\cos(\theta)\}$$

Differentiate both sides with respect to θ

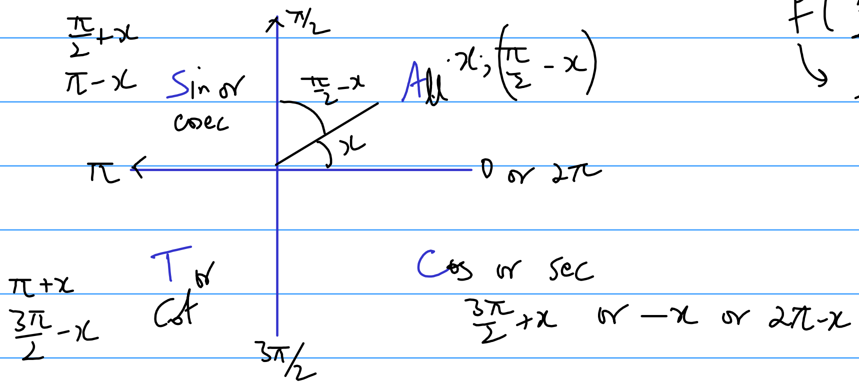
$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta}\cdot\{0+(-\sin(\theta))\} = \frac{-\sin(\theta)}{1+\cos(\theta)}$$

$$\cot\phi = \frac{-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2(\theta/2)} = -\tan\left(\frac{\theta}{2}\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \phi = \frac{\pi}{2}+\frac{\theta}{2} \Longrightarrow \text{Slope of tangent, } \tan\psi = \tan(\theta+\phi)$$

$$\tan\psi = \tan\left(\theta+\frac{\pi}{2}+\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2}+\frac{3\theta}{2}\right)$$



$$f\left(\frac{\pi}{2} \pm x\right) = \pm g(x)$$

$\hookrightarrow$ $f = \sin \Leftrightarrow g = \cos$

$f = \tan \Leftrightarrow g = \cot$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \qquad \cos\left(\frac{\pi}{2}-x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x) \qquad \text{3 more}$$

II quadrant

a) $\frac{\pi}{2} + x$

$$\sin\left(\frac{\pi}{2}+x\right) = \cos(x) \qquad \text{or} \qquad \operatorname{cosec}\left(\frac{\pi}{2}+x\right) = \sec(x)$$

$$\cos\left(\frac{\pi}{2}+x\right) = -\sin(x) \qquad \tan\left(\frac{\pi}{2}+x\right) = -\cot x$$

$$\cot\left(\frac{\pi}{2}+x\right) = -\tan(x)$$

Find the slope of the tangent at $\theta = \frac{2\pi}{3}$ on the curve $\frac{2a}{r} = 1 - \cos(\theta) \longrightarrow ①$

Ans:- slope of the tangent, $\tan\psi = \tan(\theta + \phi)$ $\hookrightarrow$?

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{d}{d\theta}\{\log(r)\}$$

$$\log\{\text{eqn } ①\} \Longrightarrow \log(2a) - \log(r) = \log\{1 - \cos(\theta)\} \longrightarrow ②$$

$$\frac{d}{d\theta}\{②\} \Longrightarrow 0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos(\theta)} \cdot \left[-\{-\sin(\theta)\}\right]$$

$$\Longrightarrow -\cot\phi = \frac{\sin(\theta)}{1-\cos(\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$\cot\phi = -\cot\left(\frac{\theta}{2}\right)$$

$$\cot\phi = \cot\left(\pi - \frac{\theta}{2}\right) \implies \phi = \pi - \frac{\theta}{2}$$

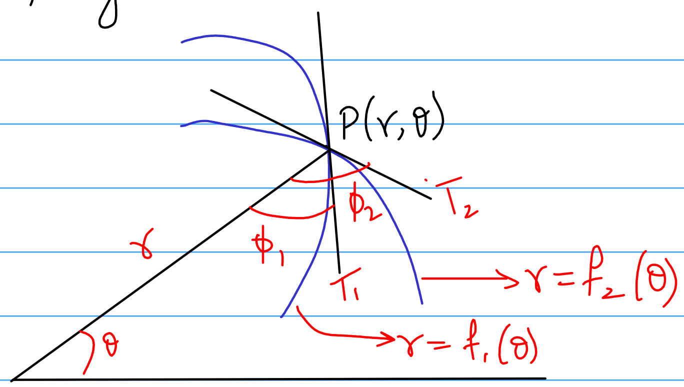
at $\theta = \frac{2\pi}{3}$ $\quad \phi\Big|_{\theta=\frac{2\pi}{3}} = \pi - \frac{1}{2}\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

Slope of tangent, $\tan\psi\Big|_{\theta=\frac{2\pi}{3}} = \tan(\theta+\phi)\Big|_{\theta=2\pi/3}$

$$= \tan\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right) = \tan\left(\frac{4\pi}{3}\right)$$

$$= \tan\left(\pi+\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Angle between curves



Angle between the curves = Angle between the tangents $= |\phi_1 - \phi_2|$

Convention: Angle between the lines = acute angle

$\therefore$ If $|\phi_1-\phi_2| > \frac{\pi}{2}$ then angle between the lines $= \pi - |\phi_1-\phi_2|$

Consider $\quad \tan|\phi_1-\phi_2| = \left|\dfrac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right| \longrightarrow (1)$

Condition for orthogonal curves

$|\phi_1-\phi_2| = \frac{\pi}{2}$

From (1) $\quad \tan(\phi_1)\tan(\phi_2) = -1 \implies \cot(\phi_1)\cot(\phi_2) = -1$

Qn) Find the angle of intersection between the curves
$r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Ans: Angle between the curves $= |\phi_1 - \phi_2|$

OR

$$\tan|\phi_1 - \phi_2| = \left| \frac{\tan\phi_1 - \tan(\phi_2)}{1 + \tan\phi_1 \tan\phi_2} \right|$$

$$r^2 \sin(2\theta) = 4 \implies \log(r^2) + \log\{\sin(2\theta)\} = \log 4$$

$$2\log r + \log\{\sin(2\theta)\} = \log 4 \longrightarrow (1)$$

$$\frac{d}{d\theta}(1) \implies 2\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\sin(2\theta)}\cos(2\theta).2 = 0$$

$$\cot\phi_1 + \cot(2\theta) = 0$$

$$\cot\phi_1 = -\cot(2\theta) = \cot(\pi - 2\theta)$$

$$\phi_1 = \pi - 2\theta \longrightarrow (2)$$

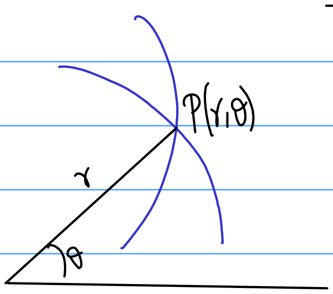
$$r^2 = 16\sin(2\theta) \implies \log(r^2) = \log(16) + \log\{\sin(2\theta)\}$$

$$2\log(r) = \log(16) + \log\{\sin(2\theta)\} \longrightarrow (3)$$

$$\frac{d}{d\theta}(3) \implies 2\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sin(2\theta)}\cos(2\theta).2$$

$$\cot\phi_2 = \cot(2\theta)$$

$$\implies \phi_2 = 2\theta \longrightarrow (4)$$



At the point of intersection of $r^2\sin(2\theta) = 4$ & $r^2 = 16\sin(2\theta)$

Substitute one in the other as r & θ value is same [see fig] at the pt of intersection

$$\{16\sin(2\theta)\}\sin(2\theta) = 4$$

$$\sin^2(2\theta) = \frac{1}{4}$$

$\implies \sin(2\theta) = \frac{1}{2}$ [$\because x^2 = a^2 \implies x = \pm a$ but
$r^2\sin(2\theta) = 4$ & $r^2 = 16\sin(2\theta) \implies \sin(2\theta) > 0$

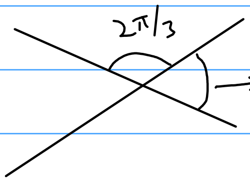
$\therefore \sin(2\theta) = -\frac{1}{2}$ is ignored [$y^2 = 4ax$, $x > 0$, same way

$$2\theta = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{12}$$

At $\theta = \frac{\pi}{12}$

$$\begin{cases} \phi_1\Big|_{\theta=\pi/2} = \pi - 2\theta\Big|_{\theta=\pi/12} = \pi - 2\left(\frac{\pi}{12}\right) = \frac{5\pi}{6} \\ \phi_2\Big|_{\theta=\pi/12} = 2\theta\Big|_{\theta=\pi/12} = 2\left(\frac{\pi}{12}\right) = \frac{\pi}{6} \end{cases}$$

$$|\phi_1 - \phi_2|\Big|_{\theta=\frac{\pi}{12}} = \left|\frac{5\pi}{6} - \frac{\pi}{6}\right| = \frac{4\pi}{6} = \frac{2\pi}{3}$$



$\rightarrow \pi - \frac{2\pi}{3} = \frac{\pi}{3}$ By convention $|\phi_1 - \phi_2| = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

Qn) Find the angle between the curves

$$r = \frac{a}{\log\theta} \text{ and } r = a\log\theta$$

Ans:- Angle between the curves $= |\phi_1 - \phi_2|$

OR

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan(\phi_1) - \tan(\phi_2)}{1 + \tan(\phi_1)\tan(\phi_2)}\right|$$

$$r = \frac{a}{\log\theta} \Rightarrow \log(r) = \log a - \log\{\log(\theta)\} \longrightarrow (1)$$

$$\frac{d}{d\theta}\{(1)\} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\log\theta}\cdot\left(\frac{1}{\theta}\right)$$

$$\cot\phi_1 = -\frac{1}{\theta\log\theta} \Rightarrow \tan\phi_1 = -\theta\log\theta \longrightarrow (2)$$

$$r = a\log\theta \Rightarrow \log(r) = \log(a) + \log\{\log\theta\} \longrightarrow (3)$$

$$\frac{d}{d\theta}\{(3)\} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\log(\theta)}\left(\frac{1}{\theta}\right)$$

$$\Rightarrow \cot\phi_2 = \frac{1}{\theta\log\theta} \Rightarrow \tan(\phi_2) = \theta\log\theta \rightarrow (4)$$

$r = \frac{a}{\log\theta}$ and $r = a\log\theta \Longrightarrow \frac{a}{\log\theta} = a\log\theta$ at the points of intersection.

$$(\log\theta)^2 = 1 \Longrightarrow \log\theta = \pm 1$$

But $r = \frac{a}{\log\theta}$ & $r = a\log\theta \Longrightarrow \log(\theta) > 0 \quad \therefore \log\theta = 1$

$$\Longrightarrow \theta = e$$

At $\theta = e$ $\quad$ (2) $\Longrightarrow \tan(\phi_1)\Big|_{\theta=e} = -e\log e = -e$

(4) $\Longrightarrow \tan\phi_2\Big|_{\theta=e} = e\log e = e$

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan(\phi_1) - \tan(\phi_2)}{1 + \tan(\phi_1)\tan(\phi_2)}\right| = \left|\frac{-e-e}{1+(-e)(e)}\right|$$

$$= \left|\frac{-2e}{1-e^2}\right| = \frac{2e}{e^2-1} \checkmark$$

$$|\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{e^2-1}\right)$$

$e > 2$
$e^2 > 4$
$e^2 - 1 > 0$

Find the angle of intersection between the curves
$r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Ans: Angle between curves $= |\phi_1 - \phi_2|$
or

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan(\phi_1) - \tan(\phi_2)}{1 + \tan(\phi_1)\tan(\phi_2)}\right|$$

$$r = \frac{a\theta}{1+\theta} \Longrightarrow \log(r) = \log(a\theta) - \log(1+\theta)$$

$$\log(r) = \log(a) + \log(\theta) - \log(1+\theta) \longrightarrow (1)$$

$$\frac{d}{d\theta}(1) \Longrightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1+\theta-\theta}{\theta(1+\theta)} = \frac{1}{\theta(1+\theta)}$$

$$\cot\phi_1 = \frac{1}{\theta(1+\theta)} \quad \text{or} \quad \tan\phi_1 = \theta(1+\theta) \longrightarrow (2)$$

Now $r = \frac{a}{1+\theta^2} \Longrightarrow \log r = \log(a) - \log(1+\theta^2) \longrightarrow (3)$

$$\frac{d}{d\theta}\{(3)\} \Longrightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{1+\theta^2}2\theta$$

$$\cot\phi_2 = \frac{-2\theta}{1+\theta^2} \Longrightarrow \tan\phi_2 = -\frac{(1+\theta^2)}{2\theta}$$

At the points of intersection, $\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$

$\theta(1+\theta^2) = 1+\theta \Longrightarrow \theta^3 = 1 \Longrightarrow \theta = 1$ [ω & ω^2 are complex and hence ignored.

$$\tan\phi_1\Big|_{\theta=1} = 1(1+1) = 2 \qquad \tan\phi_2\Big|_{\theta=1} = -\frac{(1+1)}{2} = -1$$

$$\tan|\phi_1-\phi_2| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right| = \left|\frac{2-(-1)}{1+(2)(-1)}\right| = \left|\frac{3}{-1}\right|$$

$$\tan|\phi_1-\phi_2| = 3 \Longrightarrow |\phi_1-\phi_2| = \tan^{-1}(3)$$

Show that the curves $r = 4\sec^2(\theta/2)$ and $r = 9\operatorname{cosec}^2(\theta/2)$. intersect orthogonally

Ans:- Show that $|\phi_1-\phi_2| = \frac{\pi}{2}$ or $\tan\phi_1\tan\phi_2 = -1$ or $\cot\phi_1\cot\phi_2 = -1$

$$r = 4\sec^2\left(\frac{\theta}{2}\right) \Longrightarrow \log(r) = \log(4) + \log\left(\sec^2(\theta/2)\right)$$

$$\log r = \log 4 + 2\log\left\{\sec\left(\frac{\theta}{2}\right)\right\} \longrightarrow (1)$$

$$\frac{d}{d\theta}(1) \Longrightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + 2\frac{1}{\sec(\theta/2)}\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\right)$$

$$\cot\phi_1 = \tan\left(\frac{\theta}{2}\right) \longrightarrow (2)$$

$$r = 9\,\mathrm{cosec}^2\left(\frac{\theta}{2}\right) \Rightarrow \log(r) = \log 9 + \log\left\{\mathrm{cosec}^2\left(\frac{\theta}{2}\right)\right\}$$

$$\Rightarrow \log(r) = \log 9 + 2\log\left\{\mathrm{cosec}\left(\frac{\theta}{2}\right)\right\} \longrightarrow (3)$$

$$\frac{d}{d\theta}\{(3)\} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + 2\,\frac{1}{\mathrm{cosec}\left(\frac{\theta}{2}\right)}\left\{-\mathrm{cosec}\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)\right\}\frac{1}{2}$$

$$\cot\phi_2 = -\cot\left(\frac{\theta}{2}\right)$$

At the point of intersection $\cot\phi_1 \cot\phi_2 = \tan\left(\frac{\theta}{2}\right)\left\{-\cot\left(\frac{\theta}{2}\right)\right\}$

$$= -1$$

$\therefore$ The two curves intersect orthogonally

Find the angle between the curves $r^n = a^n\cos(n\theta)$ and $r^n = b^n\sin(n\theta)$

Ans:- Angle between curves $= |\phi_1 - \phi_2|$

OR

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}\right|$$

$$r^n = a^n\cos(n\theta) \Rightarrow n\log r = \log(a^n) + \log\{\cos(n\theta)\} \longrightarrow (1)$$

$$\frac{d(1)}{d\theta} \Rightarrow \cancel{n}\,\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos(n\theta)}\{-\sin(n\theta)\,\cancel{n}\}$$

$$\cot\phi_1 = -\tan(n\theta) \longrightarrow (2)$$

$$\left[\log\{\cos(n\theta)\} = \log\{x\}\right.$$

$$\frac{d}{d\theta}\{\log x\} = \frac{1}{x}\frac{dx}{d\theta}$$

$$r^n = b^n\sin(n\theta)$$

$$\Rightarrow n\log(r) = \log b^n + \log\{\sin(n\theta)\} \longrightarrow (3)$$

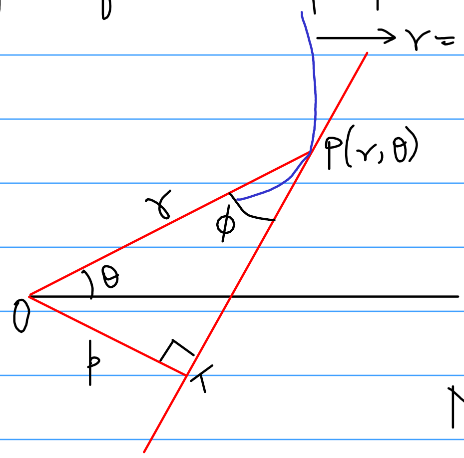
$$\frac{d(3)}{d\theta} \Rightarrow \cancel{n}\,\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin(n\theta)}\{\cos(n\theta)\cdot\cancel{n}\}$$

$$\cot\phi_2 = \cot(n\theta) \longrightarrow (4)$$

From (2) and (4) $\cot\phi_1 \cot\phi_2 = -\tan(n\theta)\cot(n\theta) = -1$

$\Rightarrow$ The two curves intersect orthogonally $\Rightarrow |\phi_1 - \phi_2| = \frac{\pi}{2}$

Length of the perpendicular from pole to the tangent



$\overline{OT} = p = ?$

From Δ OPT

$p = r\sin(\phi) \longrightarrow (1)$

Need to use $\tan\phi$ or $\cot\phi$

$$p^2 = r^2\sin^2\phi \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\cot^2\phi\right\} = \frac{1}{r^2}\left\{1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right\} \longrightarrow (2)$$

Qn) Find the length of the perpendicular from the pole to the tangent to the curve $r = a\sec^2(\theta/2)$ at $\theta = \pi/3$

Ans:- We need to find p.

$$p = r\sin\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}\left\{1+\cot^2\phi\right\}$$

$$r = a\sec^2(\theta/2) \Rightarrow \log(r) = \log(a) + \log\left\{\sec^2(\theta/2)\right\}$$

$$\Rightarrow \log(r) = \log(a) + 2\log\left\{\sec(\theta/2)\right\} \longrightarrow (1)$$

$$\frac{d}{d\theta}\{(1)\} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + 2\frac{1}{\sec(\theta/2)}\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\frac{1}{2}$$

$$\cot\phi = \tan\left(\frac{\theta}{2}\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\tan^2\left(\frac{\theta}{2}\right)\right\} = \frac{1}{r^2}\sec^2(\theta/2)$$

$$\boxed{p^2 = r^2\cos^2(\theta/2)}$$

$\rightarrow$ at $\theta = \frac{\pi}{3}$ $\quad r = a\sec^2\left(\frac{\pi/3}{2}\right) = a\sec^2\left(\frac{\pi}{6}\right) = a\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4a}{3}$

$$p^2\Big|_{\theta=\pi/3} = \left(\frac{4a}{3}\right)^2\cos^2\left(\frac{\pi/3}{2}\right) = \left(\frac{4a}{3}\right)^2\cos^2\left(\frac{\pi}{6}\right) = \left(\frac{4a}{3}\right)^2\left(\frac{\sqrt{3}}{2}\right)^2$$

$$p_{\theta=\pi/3} = \frac{4a}{3}\frac{\sqrt{3}}{2} = \frac{2a}{\sqrt{3}}$$

Find the length of the perpendicular from the pole to the tangent to the curve $r = a(1+\cos\theta)$ at $\theta = \pi/2$.

Ans $\quad p = r\sin(\phi)$ or $\frac{1}{p^2} = \frac{1}{r^2}\{1+\cot^2\phi\}$

$$\log(r) = \log(a) + \log\{1+\cos(\theta)\} \longrightarrow (1)$$

$$\frac{d}{d\theta}\{(1)\} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta}\{-\sin\theta\} = -\frac{2\sin\left(\frac{\theta}{2}\right)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$$\cot\phi = -\tan\left(\frac{\theta}{2}\right)$$

at $\theta = \frac{\pi}{2}$ $\quad \cot\phi = -\tan\left(\frac{\pi/2}{2}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$

at $\theta = \pi/2$ $\quad r = a\left(1+\cos\frac{\pi}{2}\right) \Rightarrow r = a$

$$\frac{1}{p^2}\Big|_{\theta=\pi/2} = \frac{1}{a^2}\{1+(-1)^2\} = \frac{2}{a^2}$$

$$p^2 = \frac{a^2}{2} \quad \text{or} \quad p = \frac{a}{\sqrt{2}}$$

Alternate representation.

→ Pedal equation of a polar curve

A relation between p and r only {No θ is involved} is called pedal equation (p-r equation) [OR]

[We can have pedal eqn for $y=f(x)$ also]

Obtain the pedal equation of $r=a\{1-\cos(\theta)\}$

Ans: We have $\boxed{p=r\sin\phi}$ or $\frac{1}{p^2}=\frac{1}{r^2}\{1+\cot^2\phi\}$

$$r=a(1-\cos(\theta)) \longrightarrow (1)$$

$$\Longrightarrow \log(r)=\log(a)+\log\{1-\cos(\theta)\} \rightarrow (2)$$

$$\frac{d}{d\theta}\{(2)\} \Longrightarrow \frac{1}{r}\frac{dr}{d\theta}=0+\frac{1}{1-\cos(\theta)}\left[-\{-\sin(\theta)\}\right]=\frac{\sin(\theta)}{1-\cos(\theta)}$$

$$\cot\phi=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}=\cot\left(\frac{\theta}{2}\right)$$

$$\phi=\frac{\theta}{2}$$

$$p=r\sin\phi \Longrightarrow p=r\sin(\theta/2) \longrightarrow (3)$$

Using (1) in (3) eliminate θ [$\times$ in finding pedal equation

$$(1) \Longrightarrow r=a\,2\sin^2(\theta/2) \Longrightarrow \sin^2(\theta/2)=\frac{r}{2a} \rightarrow (4)$$

$$\text{Square}\{(3)\} \Longrightarrow p^2=r^2\sin^2(\theta/2)$$

using (4) $$p^2=r^2\left(\frac{r}{2a}\right)=\frac{r^3}{2a}$$

$$\boxed{p^2=\frac{r^3}{2a}}$$

Find the pedal equation of

a) $r = ae^{\theta \cot \alpha}$.

$$r = ae^{\theta \cot(\alpha)} \longrightarrow (1)$$ [a & α are constants, $r = f(\theta)$

$$p = r\sin\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}\{1+\cot^2\phi\}$$

$$\log(r) = \log a + \log e^{\theta\cot(\alpha)}$$

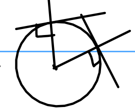
$$= \log a + \theta\cot(\alpha)$$ [$\because \log e = 1$

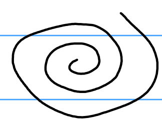
$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \cot(\alpha) \Rightarrow \cot\phi = \cot(\alpha)$$

$$\phi = \alpha$$ [tangent makes the same angle α with radius vector

$p = r\sin(\alpha)$ is the pedal equation

Eg:-





Obtain the pedal equation of $\frac{l}{r} = 1 + e\cos\theta$

l = l 1. = one

Ans:- $p = r\sin\phi$ or $\frac{1}{p^2} = \frac{1}{r^2}\{1+\cot^2\phi\}$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right\}$$

$$\frac{l}{r} = 1 + e\cos(\theta) \longrightarrow (1)$$

$$\frac{d}{d\theta}\{(1)\} \Rightarrow l\left\{-\frac{1}{r^2} \times \frac{dr}{d\theta}\right\} = e\{-\sin\theta\}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{er\sin(\theta)}{l}$$

$$\log l - \log r = \log(1 + e\cos\theta)$$

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+e\cos\theta}\left\{e\{-\sin\theta\}\right\}$$

$$\cot\phi = \frac{+e\sin(\theta)}{1+e\cos\theta} = \frac{e\sin(\theta)}{(l/r)} = \frac{er}{l}\sin(\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\cot^2\phi\right\}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left\{1+\left(\frac{er}{l}\sin\theta\right)^2\right\} = \frac{1}{r^2}\left[1+\frac{e^2r^2}{l^2}\sin^2\theta\right] \longrightarrow (2)$$

$$(1) \Rightarrow \quad \frac{l}{r} - 1 = e\cos\theta \Longrightarrow \cos\theta = \frac{1}{e}\left(\frac{l}{r}-1\right) \longrightarrow (3)$$

$$(2) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{e^2r^2}{l^2}(1-\cos^2\theta)\right]$$

Using (3) $$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{e^2r^2}{l^2} - \frac{e^2r^2}{l^2}\frac{1}{e^2}\left(\frac{l}{r}-1\right)^2\right]$$

$$= \frac{1}{r^2}\left[1+\frac{e^2r^2}{l^2} - \frac{r^2}{l^2}\left(\frac{l}{r}-1\right)^2\right]$$

Obtain the pedal equation of $r^m\cos(m\theta) = a^m$

$$r^m\cos(m\theta) = a^m \longrightarrow (1)$$

$$m\log r + \log\{\cos(m\theta)\} = m\log a$$

$$m\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos(m\theta)}\left\{-\sin(m\theta)\, m\right\} = 0$$

$$\cot\phi = \tan(m\theta) = \cot\left(\frac{\pi}{2}-m\theta\right)$$

$$\phi = \frac{\pi}{2} - m\theta$$

$$p = r\sin\phi \Longrightarrow p = r\sin\left(\frac{\pi}{2}-m\theta\right)$$

$$\frac{1}{p^2} = \frac{r^{2m-2}}{a^{2m}}$$

$$p = \frac{a^m}{r^{m-1}}$$

$$pr^{m-1} = a^m$$

$$p = r\cos(m\theta)$$

using ① $\quad p = r\left(\frac{a^m}{r^m}\right) \implies p = a^m r^{1-m}$

Obtain the pedal equation of

$$a\theta = \sqrt{r^2-a^2} - a\cos^{-1}\left(\frac{a}{r}\right) \longrightarrow ①$$

Ans:- Not advisable to take log on both sides

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} \quad \text{or} \quad \boxed{\tan\phi = r\frac{d\theta}{dr}}$$ better

$$\frac{d}{dr}\{①\} \implies a\frac{d\theta}{dr} = \frac{1}{2\sqrt{r^2-a^2}}2r - a\frac{-1}{\sqrt{1-\left(\frac{a}{r}\right)^2}}\cdot\left(-\frac{a}{r^2}\right)$$

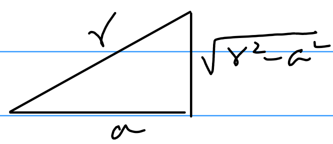
$$a\frac{d\theta}{dr} = \frac{r}{\sqrt{r^2-a^2}} - a^2\frac{r}{\sqrt{r^2-a^2}}\left(\frac{1}{r^2}\right)$$

$$a\frac{d\theta}{dr} = \frac{r}{\sqrt{r^2-a^2}}\left\{1 - \frac{a^2}{r^2}\right\} = \frac{r(r^2-a^2)}{r^2\left(\sqrt{r^2-a^2}\right)}$$

$$= \frac{\sqrt{r^2-a^2}}{r}$$

$$r\frac{d\theta}{dr} = \frac{1}{a}\sqrt{r^2-a^2}$$

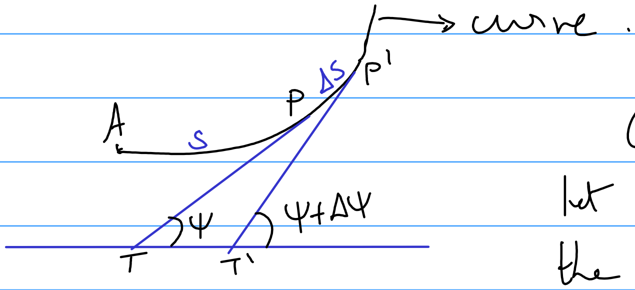
$$\therefore \tan\phi = \frac{1}{a}\sqrt{r^2-a^2}$$



$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\left(\frac{a}{\sqrt{r^2-a^2}}\right)^2\right]$$

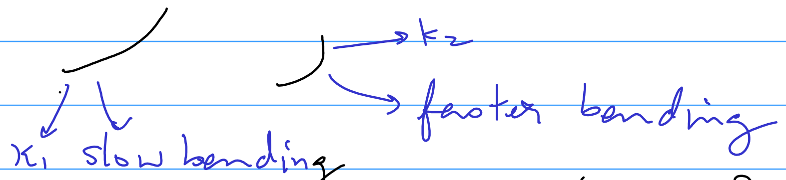
Curvature and radius of curvature



Consider a curve
let A, P, P' be any point the curve [P' is neighbouring point]
$\overset{\frown}{AP} = s \qquad \overset{\frown}{PP'} = \Delta s$

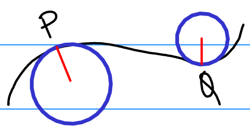
PT & PT' are tangents to the curve at P & P' making angle ψ and $\psi + \Delta\psi$ with x-axis (or initial line)

Definition;
Curvature, κ is defined as $\kappa = \dfrac{d\psi}{ds}$
↳ kappa



$\kappa_1 < \kappa_2$

Curvature of a straight line? Zero



At point P, curvature $= \kappa_1$
" " Q, " $= \kappa_2$

$\kappa_1 < \kappa_2$

Radius of curvature, $\rho = \dfrac{1}{\kappa} = \dfrac{ds}{d\psi}$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{|y_2|} \qquad y_1 \text{ is finite}$$

If $y_1 \to \infty$ $\quad \rho = \dfrac{\{1 + x_1^2\}^{3/2}}{|x_2|} \qquad x_1 = \dfrac{dx}{dy} \to 0 \quad x_2 = \dfrac{d^2x}{dy^2}$

Determine the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3+y^3=3axy$

Ans:- $$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} \longrightarrow (1)$$

$$y_1 = \frac{dy}{dx} = ? \qquad x^3+y^3 = 3axy \longrightarrow (2)$$

$\longrightarrow x \cdot y(x)$

$$\frac{d}{dx}\{(2)\} \Longrightarrow 3x^2+3y^2\frac{dy}{dx} = 3a\left\{y + x\frac{dy}{dx}\right\}$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax} \longrightarrow (3)$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -1 \longrightarrow (4)$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\{(3)\} \Longrightarrow y_2 = \frac{(y^2-ax)\{ay_1-2x\}-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2}$$

$$\left[\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right.$$

$$\left.y_2\right|_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{\left\{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right\}\left\{a(-1)-2\left(\frac{3a}{2}\right)\right\}-\left\{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2\right\}\left\{2\left(\frac{3a}{2}\right)(-1)-a\right\}}{\left\{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)\right\}^2}$$

$$= \frac{\left\{\frac{9a^2}{4}-\frac{3a^2}{2}\right\}\left\{(-4a)\right\} - \left\{\frac{3a^2}{2}-\frac{9a^2}{4}\right\}\left\{-4a\right\}}{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)^2} =$$

$$= \frac{(-4a)\left\{\left(\frac{3a^2}{4}\right) - \left(-\frac{3a^2}{4}\right)\right\}}{\left(\frac{3a^2}{4}\right)^2} = \frac{-8a}{(3a^2/4)} = \frac{-32}{3a} \longrightarrow (5)$$

$$\left.\rho\right|_{\left(\frac{3a}{2},\frac{3a}{2}\right)} = \frac{(1+y_1^2)^{3/2}}{|y_2|} = \frac{\{1+(-1)^2\}^{3/2}}{\left|\frac{-32}{3a}\right|} = \frac{3(2^{3/2}a)}{32} = \frac{3a\sqrt{2}}{16} = \frac{3a}{8\sqrt{2}}$$

$16 \to 8\times 2 \to \sqrt{2}\sqrt{2}$

Determine the radius of curvature at $(a,0)$ on the curve $xy^2 = a^3 - x^3$

Ans $y^2 + x\,2yy_1 = -3x^2$ $\qquad \rho = \dfrac{(1+y_1^2)^{3/2}}{|y_2|}$, $y_1 =$ finite

$2xyy_1 = -3x^2 - y^2$

$$y_1 = \frac{-3x^2-y^2}{2xy} = -\frac{(3x^2+y^2)}{2xy} \longrightarrow (1)$$

$$y_1\Big|_{(a,0)} = -\frac{\{3a^2+0\}}{0} \longrightarrow \infty$$

$$\Rightarrow \rho = \frac{(1+x_1^2)^{3/2}}{|x_2|} \rightarrow (2) \quad x_1 = \frac{dx}{dy}$$

$y_1 \to \infty$ at $(a,0)$ $\Rightarrow$ $x_1 \to 0$ at $(a,0)$

$$x_1 = \frac{dx}{dy} = \frac{1}{\text{eqn }(1)} \Rightarrow x_1 = -\frac{2xy}{3x^2+y^2} \longrightarrow (3)$$

$$x_2 = \frac{d^2x}{dy^2} = \frac{d}{dy}(\text{eqn }(3)) \qquad \left[\text{Now } x_2 \neq \frac{1}{y_2} \text{ but } x_1 = \frac{1}{y_1}\right.$$

$$x_2 = -2\left[\frac{(3x^2+y^2)\underbrace{\frac{d}{dy}}_{=*}(xy) - xy\frac{d}{dy}(3x^2+y^2)}{(3x^2+y^2)^2}\right]$$

$$x_2 = -2\left[\frac{(3x^2+y^2)\{x_1y+x\} - xy\{6xx_1+2y\}}{(3x^2+y^2)^2}\right]$$

$$x_2\Big|_{(a,0)} = -2\left[\frac{(3a^2+0)\{0+a\}-0}{(3a^2+0)^2}\right] = -2\frac{(3a^3)}{9a^4} = \frac{-2}{3a}$$

$$\rho\Big|_{(a,0)} = \frac{(1+0)^{3/2}}{\left|\frac{-2}{3a}\right|} = \frac{3a}{2}$$

Determine the radius of curvature of the curve

$y^4 + x^3 + a(x^2+y^2) - a^2 y = 0$ at the origin.

Ans:- $\rho = \dfrac{(1+y_1^2)^{3/2}}{|y_2|}$

$$4y^3 y_1 + 3x^2 + a\{2x + 2yy_1\} - a^2 y_1 = 0$$

$$(4y^3 + 2ay - a^2) y_1 = -(3x^2 + 2ax)$$

$$y_1 = \frac{-(3x^2+2ax)}{4y^3+2ay-a^2}$$

$$y_1\Big|_{(0,0)} = \frac{-0}{-a^2} = 0$$

$$y_2 = -\left[\frac{(4y^3+2ay-a^2)(6x+2a) - (3x^2+2ax)(12y^2y_1 + 2ay_1)}{(4y^3+2ay-a^2)^2}\right]$$

$$y_2\Big|_{(0,0)} = -\left[\frac{(-a^2)(2a) - 0(0)}{(-a^2)^2}\right] = -\left[\frac{-2a^3}{a^4}\right] = \frac{2}{a}$$

$$\rho\Big|_{(0,0)} = \frac{(1+0^2)^{3/2}}{\left|\frac{2}{a}\right|} = \frac{a}{2}$$

If $y = \dfrac{ax}{a+x}$ show that $\left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{2\rho}{a}\right)^{2/3}$

$$\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|} \longrightarrow (1)$$

$$y_1 = a\left[\frac{(a+x)1 - x(1)}{(a+x)^2}\right] = \frac{a^2}{(a+x)^2} = \frac{1}{x^2}\left\{\frac{a^2x^2}{(a+x)^2}\right\} = \frac{1}{x^2}\left(\frac{ax}{a+x}\right)^2$$

$$y_1 = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2 \longrightarrow (2)$$

$$y_1 = \frac{a^2}{(a+x)^2} \Rightarrow y_2 = a^2\left\{\frac{-2}{(a+x)^3}\right\} = -2\frac{a^2}{(a+x)^3}$$

$$y_2 = \frac{-2}{ax^3}\frac{a^2(ax^3)}{(a+x)^3} = \frac{-2}{ax^3}\times\left(\frac{ax}{a+x}\right)^3 = \frac{-2y^3}{ax^3} \longrightarrow (3)$$

(2) and (3) in (1)

$$\rho = \frac{\{1+(y/x)^4\}^{3/2}}{\left|-\frac{2}{a}\left(\frac{y}{x}\right)^3\right|} = \frac{\left\{1+\left(\frac{y}{x}\right)^4\right\}^{3/2}}{\frac{2}{a}\left(\frac{y}{x}\right)^3}$$

Raise to $\frac{2}{3}$ on both sides

$$\rho^{2/3} = \frac{1+(y/x)^4}{\left(\frac{2}{a}\right)^{2/3}\left[\left(\frac{y}{x}\right)^3\right]^{2/3}}$$

$$\left(\frac{2}{a}\right)^{2/3}\rho^{2/3} = \frac{1+(y/x)^4}{(y/x)^2} = \left(\frac{x}{y}\right)^2\left\{1+\left(\frac{y}{x}\right)^4\right\}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2$$

Show that radius of curvature at $\left(\frac{a}{4},\frac{a}{4}\right)$ on the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is $\frac{a}{\sqrt{2}}$

Ans: $\rho = \frac{(1+y_1^2)^{3/2}}{|y_2|}$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1 = 0 \Rightarrow y_1 = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$y_1\Big|_{\left(\frac{a}{4},\frac{a}{4}\right)} = -1$$

$$y_2 = -\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\,y_1 - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}\right] = -\left[\frac{\sqrt{\frac{x}{y}}\,y_1 - \sqrt{\frac{y}{x}}}{2x}\right]$$

$$y_2\Big|_{\left(\frac{a}{4},\frac{a}{4}\right)} = -\left[\frac{1(-1)-1}{2(a/4)}\right] = -\frac{(-2)}{a/2} = \frac{4}{a}$$

$$\rho = \frac{\left[1+(-1)^2\right]^{3/2}}{|4/a|} = 2^{3/2}\,\frac{a}{4} = 2\sqrt{2}\,\frac{a}{4} = \frac{a}{\sqrt{2}}$$

Radius of curvature of polar curves

$$\kappa = \frac{d\psi}{ds} \qquad \psi = \theta + \phi$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}$$

alternately

$$\boxed{\rho = r\,\frac{dr}{dp}}$$

Find the radius of curvature at any point on the curve $r(1+\cos\theta) = a \longrightarrow (1)$

Ans:- $\rho = r\,\underbrace{\frac{dr}{dp}}_{\text{2nd}}$ $\qquad \underbrace{p = r\sin\phi}_{\text{1st}}$ or $\frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$

$$\log r + \log\{1+\cos\theta\} = \log a \longrightarrow (2)$$

$$\left[\begin{array}{l}\log(AB) = \log A + \log B \\ \log(A+B) \neq \log A + \log B\end{array}\right.$$

$$\frac{d}{d\theta}(2) \implies \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{1+\cos(\theta)}\{-\sin\theta\} = 0$$

$$\cot\phi = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan\left(\frac{\theta}{2}\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \implies \phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$p = r\sin\phi \implies p = r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = r\cos\left(\frac{\theta}{2}\right) \longrightarrow (3)$$

$$(1) \implies r(1+\cos\theta) = a$$

$$r\,2\cos^2(\theta/2) = a \longrightarrow \cos^2\left(\frac{\theta}{2}\right) = \frac{a}{2r}$$

$$(3)^2 \implies p^2 = r^2\cos^2(\theta/2) = r^2\left(\frac{a}{2r}\right)$$

$$p^2 = \frac{ar}{2} \longrightarrow (4) \qquad \left[\because\ p = \sqrt{\frac{ar}{2}}\right.$$

$$\rho = r\frac{dr}{dp}$$

$$(4) \implies r = \frac{2p^2}{a} \implies \frac{dr}{dp} = \frac{4p}{a}$$

$$\rho = r\frac{4p}{a} = \frac{4r}{a}\sqrt{\frac{ar}{2}} = \frac{(2r)^{3/2}}{\sqrt{a}}$$

Show that radius of curvature varies inversely as r^{n-1} for the curve $r^n = a^n\cos(n\theta)$

Ans:- To show that $\rho \propto \frac{1}{r^{n-1}}$

$$\rho = r\frac{dr}{dp} \qquad p = r\sin\phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$r^n = a^n \cos(n\theta) \longrightarrow (1)$$

$$\log\{(1)\} \Longrightarrow n \log r = \log(a^n) + \log\{\cos(n\theta)\} \longrightarrow (2)$$

$$\frac{d}{d\theta}\{(2)\} \Longrightarrow n \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos(n\theta)}\{-\sin(n\theta).n\}$$

$$\cot\phi = -\tan(n\theta) = \cot\left\{\frac{\pi}{2} + n\theta\right\}$$

$$\phi = \frac{\pi}{2} + n\theta$$

$$p = r\sin\phi \Longrightarrow p = r\sin\left(\frac{\pi}{2} + n\theta\right) = r\cos(n\theta) \longrightarrow (3)$$

$$(1) \Longrightarrow \cos(n\theta) = \frac{r^n}{a^n} \Longrightarrow p = r\left\{\frac{r^n}{a^n}\right\} = \frac{r^{n+1}}{a^n}$$

$$p = \frac{r^{n+1}}{a^n} \longrightarrow (4)$$

Radius of curvature, $\rho = r\dfrac{dr}{dp} \longrightarrow (5)$

$$\frac{d}{dp}\{\text{eqn } (4)\} \Longrightarrow 1 = \frac{1}{a^n}(n+1)r^n\frac{dr}{dp} \longrightarrow \frac{dr}{dp} = \frac{a^n}{(n+1)r^n} \longrightarrow (6)$$

$$(6) \text{ in } (5) \Rightarrow \rho = r\left\{\frac{a^n}{(n+1)r^n}\right\} = \frac{a^n}{(n+1)r^{n-1}}$$

$$\rho \propto \frac{1}{r^{n-1}}$$

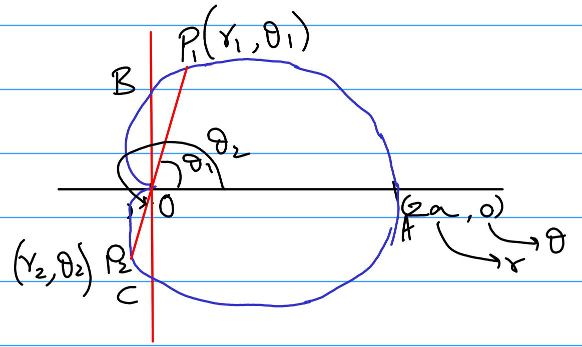
If ρ_1 and ρ_2 are the radii of curvature at the extremities of a chord through the pole for the polar curve $r = a(1+\cos\theta)$, prove that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

$\longrightarrow$ Cardioid

Ans:-

θ	0	$\pi/2$	π	$3\pi/2$	2π
r	$2a$	a	0	a	$2a$

$f(-\theta) = f(\theta) \Rightarrow$ Curve is symmetric about initial line



OB is a chord through pole
OA " " " "
OC " " " "
P_1P_2 " " " "

$r_1 \neq r_2$

$\theta_2 = \angle AOP_1 + \angle P_1OP_2$

$\theta_2 = \theta_1 + \pi$

$\rho = ?$ $\quad \rho_1 = \rho(\text{at } P_1) \quad \rho_2 = \rho(\text{at } P_2)$

$\underbrace{\rho}_{4^{th}} = r\underbrace{\frac{dr}{dp}}_{3^{rd}} \qquad \underbrace{p = r\sin\phi}_{2^{nd}}$ or $\underbrace{\frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]}_{1^{st}}$

$$r = a(1+\cos(\theta)) \longrightarrow (1)$$

$$\log(r) = \log(a) + \log\{1+\cos(\theta)\}$$

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta}\{-\sin\theta\}$$

$$\cot\phi = -\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = -\tan\left(\frac{\theta}{2}\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Longrightarrow \phi = \frac{\pi}{2}+\frac{\theta}{2}$$

$$p = r\sin\phi \Longrightarrow p = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = r\cos\left(\frac{\theta}{2}\right) \longrightarrow (2)$$

$$(1) \Longrightarrow r = a\,2\cos^2(\theta/2) \Longrightarrow \cos^2\left(\frac{\theta}{2}\right) = \frac{r}{2a} \longrightarrow (3)$$

$$\left[\text{Eqn } (2)\right]^2 \Longrightarrow \quad p^2 = r^2\cos^2\left(\frac{\theta}{2}\right)$$

$$p^2 = r^2\left(\frac{r}{2a}\right) \qquad \left[\because \text{ using eqn } (3)\right.$$

$$p^2 = \frac{r^3}{2a} \longrightarrow (4)$$

$$\frac{d}{dp}\left\{\text{eqn } (4)\right\} \Longrightarrow \quad 2p = \frac{1}{2a}\, 3r^2 \frac{dr}{dp} \Longrightarrow \frac{dr}{dp} = \frac{4ap}{3r^2}$$

$$\rho = r\frac{dr}{dp} = r\left(\frac{4ap}{3r^2}\right)$$

$$\rho = \frac{4a}{3r}\left(\sqrt{\frac{r^3}{2a}}\right)$$

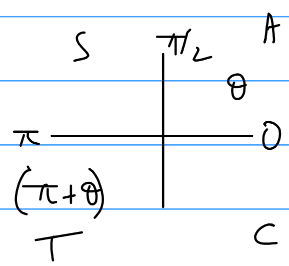
$$\rho^2 = \frac{16a^2}{9r^2}\left(\frac{r^3}{2a}\right) = \frac{8ar}{9} = \frac{8a}{9}\left\{a(1+\cos\theta)\right\}$$

$$\rho^2 = \frac{8a^2}{9}\left\{1+\cos\theta\right\} \longrightarrow (5)$$

At $P_1(r_1, \theta_1)$, $\quad \rho_1^2 = \dfrac{8a^2}{9}\left\{1+\cos(\theta_1)\right\} \longrightarrow (6)$

At $P_2(r_2, \pi+\theta_1)$, $\quad \rho_2^2 = \dfrac{8a^2}{9}\left\{1+\cos(\pi+\theta_1)\right\}$

$$\rho_2^2 = \frac{8a^2}{9}\left\{1-\cos(\theta_1)\right\} \longrightarrow (7)$$



Eqns (6) + (7) $\Longrightarrow \rho_1^2+\rho_2^2 = \dfrac{8a^2}{9}\left\{1+\cos\theta_1\right\} + \dfrac{8a^2}{9}\left\{1-\cos\theta_1\right\}$

$$\rho_1^2+\rho_2^2 = \frac{16a^2}{9}$$

Rolle's theorem

i) $f(x)$ is continuous in $[a,b]$
ii) " " differentiable in (a,b)
iii) $f(a)=f(b)$

$\longrightarrow \exists\, c\in(a,b) \mid f'(c)=0$

Lagrange's Mean value theorem (LMVT)

i) $f(x)$ is continuous in $[a,b]$
ii) " " differentiable in (a,b)

$$\Rightarrow \exists\, c\in(a,b) \mid f'(c)=\frac{f(b)-f(a)}{b-a}$$

In LMVT $b=x$

$$f'(c)=\frac{f(x)-f(a)}{x-a}$$

$$f(x)=f(a)+(x-a)f'(c) \longrightarrow (1)$$

If $x-a$ is small $c\longrightarrow a$
then (1) gives a good approximation to $f(x)$

Generalized mean value theorem [Taylor's theorem]

If $f(x), f'(x) \ldots\ldots f^{(n-1)}(x)$ are continuous in $[a,b]$ and $f^{(n)}(x)$ exists in (a,b) then there exists $c\in(a,b)$ such that

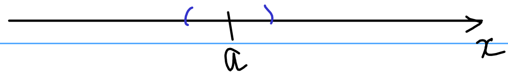
$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\ldots\ldots+R_n$$

$$R_n=\frac{(x-a)^n f^n(c)}{n!} \longrightarrow \text{Lagrange's form of remainder}$$

Taylor's theorm$\Big|_{n=1}$ = Lagrange's mean value theorem

If $R_n \to 0$ as $n \to \infty$ then the series converges to $f(x)$. This series is called the Taylor's series

Taylor's series $x=a$



$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots\cdots$$

If $a \to 0$ then we get Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\cdots$$

Expand $f(x) = 2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$.

Ans:- Taylor's series of $f(x)$ is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

powers of $(x-a)$

$$f(x) = \underbrace{f(2)} + (x-2)\underbrace{f'(2)} + \frac{(x-2)^2}{2!}\underbrace{f''(2)} + \frac{(x-2)^3}{3!}\underbrace{f'''(2)} + \cdots$$

find

$f(x) = 2x^3 + 7x^2 + x - 6$

$f'(x) = 6x^2 + 14x + 1$

$f''(x) = 12x + 14$

$f'''(x) = 12$

$f^{iv}(x) = 0 \quad f^{v}(x) = 0 \cdots\cdots$

find

$f(2) = 40$

$f'(2) = 53$

$f''(2) = 38$

$f'''(2) = 12$

substitute

$$f(x) = 40 + (x-2)(53) + \frac{(x-2)^2}{2} 38 + \frac{(x-2)^3}{6}(12)$$

$$f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

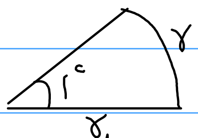
Calculate the approximate value of $\cos 32^0$ using Taylor's series.

Ans: $f(x) = \cos(x)$

Taylor's series of $f(x)$ is

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

$a \longrightarrow \underline{\underline{30^0}}$



Radian measure

Trigonometric functions $\Longrightarrow$ domain [i.e. θ] is radians

$a \longrightarrow \cancel{30^0}$ $\frac{\pi}{6}$ radians

$$f(x) = f\left(\frac{\pi}{6}\right) + \frac{\left(x-\frac{\pi}{6}\right)}{1!}f'\left(\frac{\pi}{6}\right) + \frac{\left(x-\pi/6\right)^2}{2!}f''\left(\frac{\pi}{6}\right) + \cdots$$

$$\frac{\pi}{6} \longrightarrow \frac{180^0}{6} = 30^0$$

$f(x) = \cos(x)$ $\qquad$ $f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f'(x) = -\sin(x)$ $\qquad$ $f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f''(x) = -\cos(x)$ $\qquad$ $f''\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f'''(x) = \sin(x)$ $\qquad$ $f'''\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$f(x) = \frac{\sqrt{3}}{2} + \left(x-\frac{\pi}{6}\right)\left(-\frac{1}{2}\right) + \frac{\left(x-\pi/6\right)^2}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\left(x-\frac{\pi}{6}\right)^3}{6}\left(\frac{1}{2}\right) + \cdots$$

$$f(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x-\frac{\pi}{6}\right)^3 + \cdots \quad (1)$$

$\cos 32^\circ$

$180^\circ \longrightarrow \pi$
$32^\circ \longrightarrow \frac{32\pi}{180}$ radians

$x = \frac{32\pi}{180}$ in ① then $x - \frac{\pi}{6} = \frac{32\pi}{180} - \frac{\pi}{6} = \frac{2\pi}{180} = \frac{\pi}{90}$

$$f\left(\frac{32\pi}{180}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{90}\right) - \frac{\sqrt{3}}{4}\left(\frac{\pi}{90}\right)^2 + \frac{1}{12}\left(\frac{\pi}{90}\right)^3 + \cdots\cdots$$

$\cos\left(\frac{32\pi}{180}\right) = 0.8480480428$

[Direct calculator value of $\cos(32^\circ) = 0.8480480962$

$\sqrt{3} \div 2 - \pi \div 180 - \sqrt{3} \times (\pi \div 90)^2 \div 4 + (\pi \div 90)^3 \div 12$

Evaluate log1.1 correct to four decimal places. using Taylor's series

Casio
fx991EX
fx991ES

$\log_e(1.1) = \ln(1.1) = 0.0953101798$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots$$

$x = 1.1 \qquad a = 1 \qquad f(x) = \log_e x$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots\cdots$$

$f(x) = \log_e(x)$ $\qquad$ $f(1) = \log_e 1 = 0$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(1) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad$ $f''(1) = -\frac{1}{1} = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad$ $f'''(1) = \frac{2}{1^3} = 2$

$f^{IV}(x) = \frac{2(-3)}{x^4}$ $\qquad$ $f^{IV}(1) = \frac{-6}{1^4} = -6$

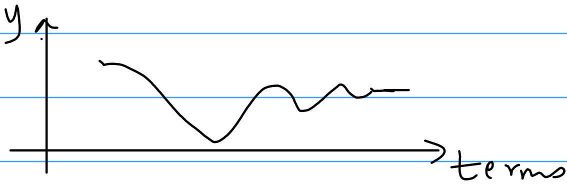
$$f(x) = \underbrace{\log x}_{\Rightarrow \log_e} = 0 + (x-1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$$

$$\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

$$x = 1.1 \Longrightarrow x - 1 = 0.1$$

$$\log_e(1.1) = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 + \cdots$$

$$= 0.\underline{095}30833333$$



Expand $\tan^{-1} x$ in powers of $(x-1)$ up to four terms.

Ans:- $f(x) = f(1) + \frac{(x-1)}{1!}f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$

$y = \tan^{-1}(x)$ $\qquad y(1) = \frac{\pi}{4} \longrightarrow 1$

$y_1 = \frac{1}{1+x^2}$ $\qquad y_1(1) = \frac{1}{1+1} = \frac{1}{2} \longrightarrow 2$

$y_2 = \frac{-1}{(1+x^2)^2}(2x) = -2xy_1^2$ $\qquad y_2(1) = -2(1)\left(\frac{1}{2}\right)^2 = -\frac{1}{2} \longrightarrow 3$

$y_3 = -2\left[y_1^2 + x(2y_1)y_2\right]$ $\qquad y_3(1) = -2\left[\left(\frac{1}{2}\right)^2 + 1(2)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right]$

$y_3(1) = -2\left[\frac{1}{4} - \frac{1}{2}\right] = \frac{1}{2} \longrightarrow 4$

$$f(x) = \frac{\pi}{4} + (x-1)\left(\frac{1}{2}\right) + \frac{(x-1)^2}{2}\left(-\frac{1}{2}\right) + \frac{(x-1)^3}{6}\left(\frac{1}{2}\right) + \cdots$$

$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$$

$$\left.\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned}\right\}$$ Maclaurin's series

Find the Maclaurin's series of $\sin(x)$.

Ans:- $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

$f(x) = \sin(x)$ $\qquad$ $f(0) = 0$

$f'(x) = \cos(x)$ $\qquad$ $f'(0) = 1$

$f''(x) = -\sin(x)$ $\qquad$ $f''(0) = 0$

$f'''(x) = -\cos(x)$ $\qquad$ $f'''(0) = -1$

$f^{iv}(x) = \sin(x)$ $\qquad$ $f^{iv}(0) = 0$

$f^{v}(x) = \cos(x)$ $\qquad$ $f^{v}(0) = 1$

$$f(x) = 0 + x(1) + \frac{x^2}{2}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \cdots$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

Similarly you can verify Maclaurin's series of $\cos(x)$, $\underbrace{e^x}_{①}$, $\underbrace{e^{-x}}_{②}$, $\cosh(x) = \frac{① + ②}{2}$, $\sinh(x) = \frac{① - ②}{2}$

Obtain the Maclaurin's series of $f(x) = \tan(x)$

Ans:- Maclaurin's series of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$y = f(x) = \tan(x)$ $\quad$ $y(0) = \tan(0) = 0$

$y_1 = \sec^2(x)$ $\quad$ $y_1(0) = \sec^2(0) = 1$

$= 1 + \tan^2(x) = 1 + y^2$ $\quad$ $y_1(0) = 1 + 0 = 1 \longrightarrow (1)$

$y_2 = 2\sec(x)\{\sec(x)\tan(x)\} = 2\sec^2(x)\tan(x)$ $\quad$ $y_2(0) = 0$

$= 2\underbrace{y_1}_{u}\underbrace{y}_{v}$ $\quad$ $y_2(0) = 2(1)(0) = 0$

$y_3 = 2[y_2 y + y_1 y_1] = 2[y_2 y + y_1^2]$ $\quad$ $y_3(0) = 2[0(0) + 1^2] = 2 \longrightarrow (2)$

$u'v + uv'$

$y_4 = 2[(y_3 y + y_2 y_1) + 2y_1 y_2]$

$= 2[y_3 y + 3y_1 y_2]$ $\quad$ $y_4(0) = 2[2(0) + 3(1)0] = 0$

$y_5 = 2[(y_4 y + y_3 y_1) + 3(y_2 y_2 + y_1 y_3)]$

$= 2[y_4 y + 4y_1 y_3 + 3y_2^2]$ $\quad$ $y_5(0) = 2[0(0) + 4(1)(2) + 3(0)^2]$

$y_5(0) = 16$

$$f(x) = \tan(x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \cdots$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Q.) Obtain the Maclaurin's series of $\log\sqrt{\frac{1+x}{1-x}}$

Ans:- Maclaurin's series of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right] \longrightarrow (1)$$

$f(x) = \log(1+x)$ $\quad$ then $\quad$ $\log(1-x) = f(-x)$

Obtain Maclaurin's series of $\log(1+x)$ then replace x by $-x$ for Maclaurin's series of $\log(1-x)$

$$y = f(x) = \log(1+x) \qquad f(0) = \log(1) = \log_e(1) = \ln(1) = 0$$

$$y_1 = \frac{1}{1+x} \qquad y_1(0) = \frac{1}{1+0} = 1$$

$$y_2 = \frac{-1}{(1+x)^2} \qquad y_2(0) = \frac{-1}{(1+0)^2} = -1$$

$$y_3 = \frac{(-1)(-2)}{(1+x)^3} = \frac{2}{(1+x)^3} \qquad y_3(0) = \frac{2}{(1+0)^3} = 2$$

$$y_4 = \frac{-6}{(1+x)^4} \qquad y_4(0) = -6$$

$$f(x) = \log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} - \cdots \longrightarrow (2)$$

$$f(x) = \log(1+x) \longrightarrow \log(1-x) = f(-x)$$

Replace x by $(-x)$ in (2)

$$\log(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \frac{(-x)^6}{6} \cdots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \cdots \longrightarrow (3)$$

(2) & (3) in eqn (1)

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} \cdots\right)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots\right]$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

Obtain the Maclaurin series of $f(x) = \log \sec x$ upto x^6.

Ans

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$y = \log(\sec x)$ $\qquad$ $y(0) = \log\{\sec(0)\} = \log 1 = 0$

$y_1 = \dfrac{1}{\sec(x)} \sec(x)\tan(x) = \tan(x)$ $\qquad$ $y_1(0) = 0$

$y_2 = \sec^2(x) = 1 + \tan^2(x) = 1 + y_1^2$ $\qquad$ $y_2(0) = 1 + \{y_1(0)\}^2 = 1$

$y_3 = (2y_1) y_2$ $\qquad$ $y_3(0) = 2(0)(1) = 0$

$y_4 = 2\left[(y_2)y_2 + y_1(y_3)\right]$ $\qquad$ $y_4(0) = 2\left[1^2 + 0(0)\right] = 2$

$y_4 = 2\left[y_2^2 + y_1 y_3\right]$

$y_5 = 2\left[(2y_2)y_3 + y_2 y_3 + y_1 y_4\right]$

$= 2\left[3y_2 y_3 + y_1 y_4\right]$ $\qquad$ $y_5(0) = 2\left[3(1)(0) + (0)(2)\right] = 0$

$y_6 = 2\left[3\{y_3^2 + y_2 y_4\} + y_2 y_4 + y_1 y_5\right]$

$y_6 = 2\left[3y_3^2 + 4y_2 y_4 + y_1 y_5\right]$ $\qquad$ $y_6(0) = 2\left[3(0)^2 + 4(1)(2) + (0)(0)\right]$

$y_6(0) = 16$

$$y = \log\{\sec(x)\} = 0 + x\{0\} + \frac{x^2}{2!}\{1\} + \frac{x^3}{3!}\{0\} + \frac{x^4}{4!}\{2\} + \frac{x^5}{5!}\{0\} + \frac{x^6}{6!}(16)$$

$$= \frac{x^2}{2!} + \frac{2(x^4)}{4!} + \frac{16x^6}{6!} + \ldots\ldots$$

Qn) Obtain the Maclaurin series of $\log(1+e^x)$

Ans:- $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f'''(0) + \ldots\ldots$

$$y = \log(1+e^x) \qquad y(0) = \log(1+e^0) = \log(2) = 0.69315$$

Not necessary

$\ln(2) = \log_e(2)$

$$y_1 = \left(\frac{1}{1+e^x}\right)e^x \qquad y_1(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$$

$$y_2 = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} = \frac{1}{e^x}\,\frac{(e^x)^2}{(1+e^x)^2} = y_1^2 e^{-x}$$

$$y_2 = y_1^2 e^{-x} \qquad y_2(0) = \left(\frac{1}{2}\right)^2 e^0 = \frac{1}{4}$$

$$y_3 = (2y_1y_2)e^{-x} + y_1^2(-e^{-x})$$
$$= 2y_1y_2e^{-x} - y_2 \qquad y_3(0) = 2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)e^0 - \frac{1}{4} = 0$$

$$y_4 = 2\left[y_2(y_2e^{-x}) + y_1y_3e^{-x} + y_1y_2(-e^{-x})\right] - y_3$$

$$y_4(0) = 2\left[\left(\frac{1}{4}\right)^2 e^0 + \frac{1}{2}(0)e^0 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)e^0\right] - 0 = 2\left(-\frac{1}{16}\right) = -\frac{1}{8}$$

$$f(x) = \log(2) + x\left\{\frac{1}{2}\right\} + \frac{x^2}{2!}\left\{\frac{1}{4}\right\} + \frac{x^3}{3!}\{0\} + \frac{x^4}{4!}\left\{-\frac{1}{8}\right\} + \cdots$$

HW Obtain the Maclawrin series of $\frac{e^x}{1+e^x}$ upto x^3

[Hint $y(0) = \frac{1}{2}$ $y_1(0) = \frac{1}{4}$ $y_2(0) = 0$ $y_3(0) = -\frac{1}{8}$]

show that $f(x) = \frac{1}{2} + x\frac{1}{4} + \frac{x^2}{2!}\{0\} + \frac{x^3}{3!}\left\{-\frac{1}{8}\right\} + \cdots$

Qn) Obtain the Maclawrin series of $\log\{1+\sin(x)\}$

Ans:- $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$$y = f(x) = \log(1+\sin(x)) \qquad f(0) = \log\{1+\sin(0)\} = 0$$

$$y_1 = \frac{1}{1+\sin(x)}\cos(x) \qquad y_1(0) = \frac{\cos(0)}{1+\sin(0)} = 1$$

$$y_1 = \frac{\cos(x)}{1 + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} = \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\frac{\left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}\left\{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right\}}{\left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}^2}$$

$$y_1 = \frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)} = \frac{1 - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)} = \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{x}{2}\right)}$$

$$y_1 = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$y_2 = \sec^2\left(\frac{\pi}{4} - \frac{x}{2}\right) \cdot \left(-\frac{1}{2}\right) \qquad y_2(0) = \sec^2\left(\frac{\pi}{4}\right)\left(-\frac{1}{2}\right) = -\frac{1}{2}\left(\sqrt{2}\right)^2 = -1$$

$$y_2 = \frac{-1}{2}\left\{1 + \tan^2\left(\frac{\pi}{4} - \frac{x}{2}\right)\right\} = -\frac{1}{2}\left\{1 + y_1^2\right\}$$

$$y_3 = -\frac{1}{2}\left\{2y_1y_2\right\} = -y_1y_2 \qquad y_3(0) = -(1)(-1) = 1$$

$$y = 0 + x(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(1) + \cdots\cdots$$

$$y = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots\cdots$$

HW Maclaurin series of $\sqrt{1 + \sin(2x)}$

{ use $\sin(2x)$ & trigonometric identity

Qn) Determine the approximate value of π using the Maclaurin's series expansion of $\sin^{-1} x$.

Ans:- $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$$y = \sin^{-1}(x) \qquad y(0) = 0$$

$$y_1 = \frac{1}{\sqrt{1 - x^2}} \qquad y_1(0) = \frac{1}{\sqrt{1 - 0}} = 1$$

$$y_1^2 = \frac{1}{1-x^2}$$

$$(1-x^2)y_1^2 = 1$$

$$(1-x^2)(2y_1y_2) + (-2x)y_1^2 = 0$$

$$(1-x^2)y_2 - xy_1 = 0 \longrightarrow (1)$$

At $x=0$ $\quad (1-0)y_2(0) - 0 = 0 \implies y_2(0) = \frac{0}{1} = 0$

diff (1) wrt x

$$\{(1-x^2)y_3 + (-2x)y_2\} - \{xy_2 + y_1\} = 0$$

$$(1-x^2)y_3 - 3xy_2 - y_1 = 0 \longrightarrow (2)$$

at $x=0$ $\quad (1-0)y_3(0) - 3(0)(0) - 1 = 0 \implies y_3(0) = 1$

diff (2) wrt x

$$(1-x^2)y_4 - 2xy_3 - 3(xy_3 + y_2) - y_2 = 0$$

$$(1-x^2)y_4 - 5xy_3 - 4y_2 = 0 \longrightarrow (3)$$

$(1)y_4(0) - 5(0)(1) - 4(0) = 0$

$\implies y_4(0) = 0$

diff (3) wrt x

$$\{(1-x^2)y_5 + (-2x)y_4\} - 5\{xy_4 + y_3\} - 4y_3 = 0$$

$$(1-x^2)y_5 - 7xy_4 - 9y_3 = 0$$

$y_5(0) = 9$

$$f(x) = 0 + \frac{x}{1!}\{1\} + \frac{x^2}{2!}\{0\} + \frac{x^3}{3!}\{1\} + \frac{x^4}{4!}\{0\} + \frac{x^5}{5!}\{9\}$$

$$\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots\cdots$$

$$x = \frac{1}{2} \implies \sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{(1/2)^3}{6} + \frac{3(1/2)^5}{40} + \cdots\cdots$$

$$\implies \frac{\pi}{6} = 0.5231771$$

$$\pi = 6(0.5231771) = 3.1390625$$

Obtain the Maclaurin's series of $e^{\sin(x)}$ up to the term containing x^4

Ans:- $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots$

$y = e^{\sin(x)}$ $\qquad$ $y(0) = e^0 = 1$

$y_1 = e^{\sin(x)} \cos(x) = y\cos(x)$ $\qquad$ $y_1(0) = y(0)(1) = 1$

$y_2 = y_1 \cos(x) + y\{-\sin(x)\}$ $\qquad$ $y_2(0) = (1)(1) - (1)0 = 1$

$y_3 = y_2 \cos(x) - y_1 \sin(x) - \{y_1 \sin(x) + y\cos(x)\}$

$\quad = y_2 \cos(x) - 2y_1 \sin(x) - y_1$ $\qquad$ $y_3(0) = (1)(1) - 2(1)(0) - 1 = 0$

$y_4 = y_3 \cos(x) - y_2 \sin(x) - 2\{y_2 \sin(x) + y_1 \cos(x)\} - y_2$

$\quad = y_3 \cos(x) - 3y_2 \sin(x) - 2y_1 \cos(x) - y_2$

$y_4(0) = (0)(1) - 3(1)(0) - 2(1)(1) - 1 = -3$

$$f(x) = 1 + \frac{x}{1!}\{1\} + \frac{x^2}{2!}\{1\} + \frac{x^3}{3!}\{0\} + \frac{x^4}{4!}\{-3\} + \cdots\cdots$$

$$f(x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + - \cdots\cdots$$

HW:- Maclaurin series of a) $f(x) = e^x \cos(x)$

b) $f(x) = x^5 \sin^{-1}(x)$