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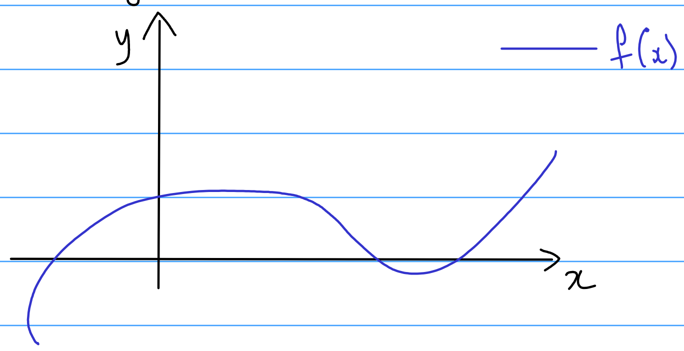
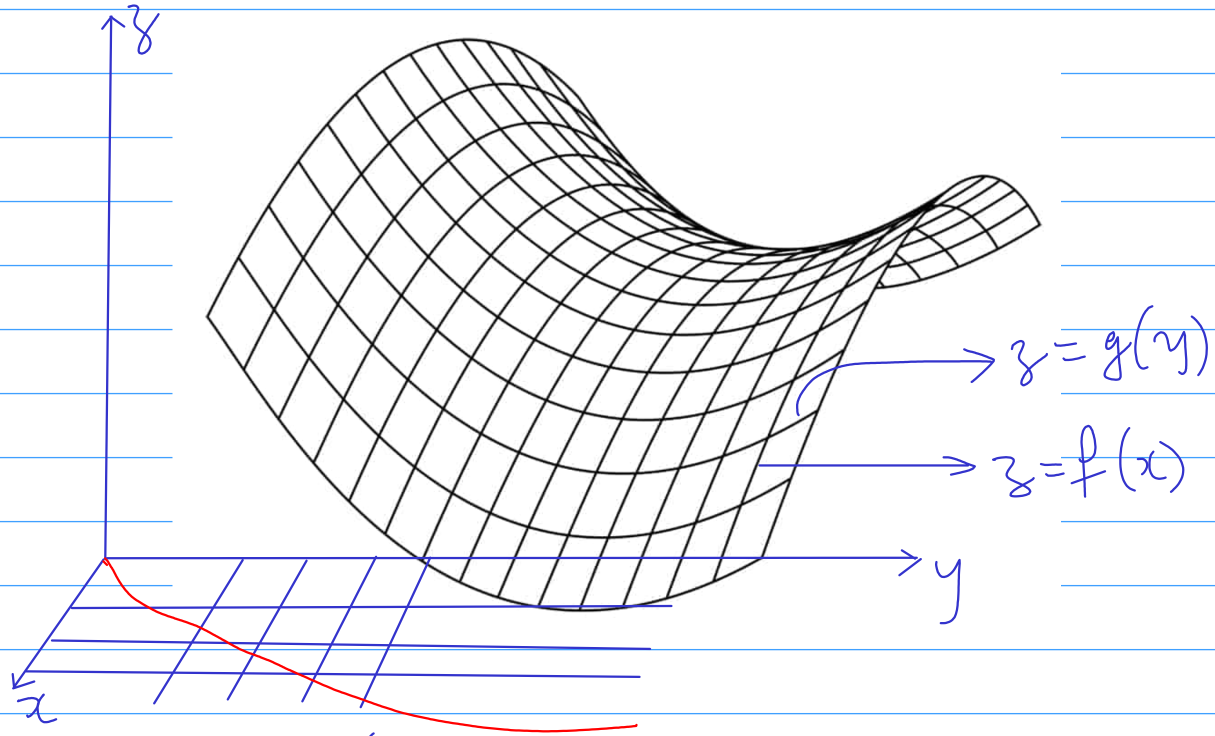
Unit 2: Differential Calculus-2

$y = f(x) \longrightarrow$ Curve in the XY-plane



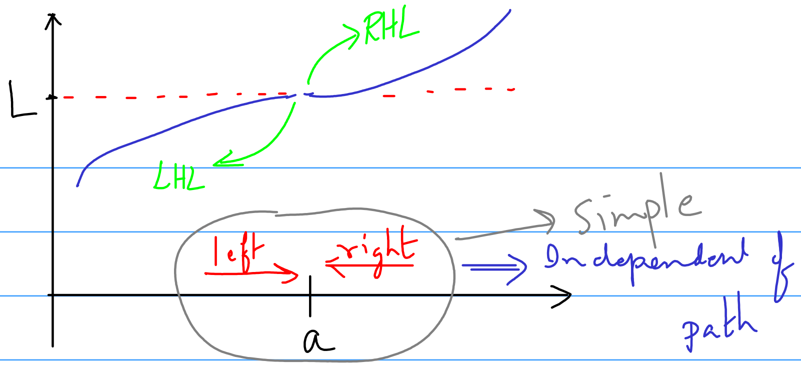


$z = F(x, y) \longrightarrow$ Surface

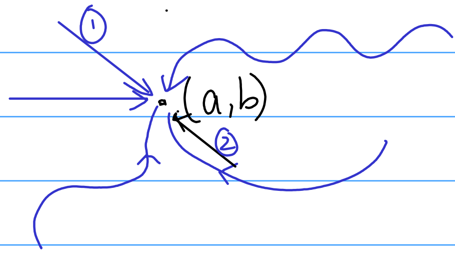
$z = F(x, k) = f(x)$

$z = F(h, y) = g(y)$

$$\lim_{x\to a} f(x) = L$$



$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$$



Domain $D \subseteq \mathbb{R}\times\mathbb{R}$ [i.e. $\mathbb{R}^2$]

$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$ only if limit is independent of the path

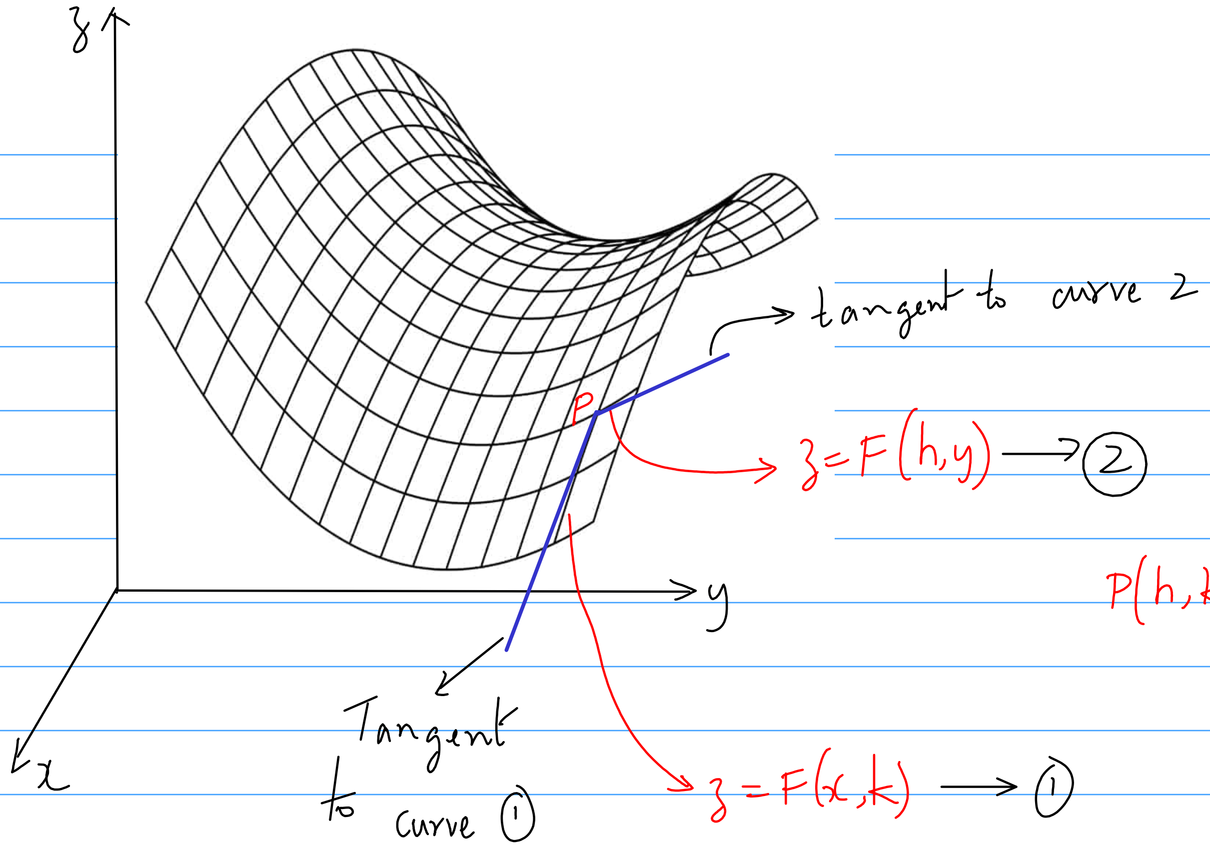
Continuity

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$$

Differentiability

$$\lim_{\delta x\to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x} = \frac{\partial f}{\partial x},$$ partial derivative of f wrt 'x' [i.e. all other variables are kept constant

$$\lim_{\delta y\to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y} = \frac{\partial f}{\partial y},$$ partial derivative of f wrt 'y' [i.e. all other variables are kept constant



$P(h,k)$

$$\left.\frac{\partial z}{\partial x}\right|_{(h,k)} = \text{slope of tangent to curve } z = F(x,k) \text{ at } (h,k)$$

Similarly $$\left.\frac{\partial z}{\partial y}\right|_{(h,k)} = \text{slope of tangent to the curve } z = F(h,y) \text{ at } (h,k)$$

$z = f(x,y)$

$y = f(x) \Rightarrow y_1 = \frac{dy}{dx}$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial f}{\partial x} = z_x = f_x \\ \frac{\partial z}{\partial y} &= \frac{\partial f}{\partial y} = z_y = f_y \end{aligned}\right\} \text{1st order partial derivatives}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = z_{xx} = f_{xx}$$

$$\left.\begin{aligned} \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) &= \frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial y \partial x} = z_{xy} = f_{xy} \quad \left[\because \frac{\partial}{\partial y}(z_x)\right] \\ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = z_{yx} = f_{yx} \quad \left[\because \frac{\partial}{\partial x}(z_y)\right] \end{aligned}\right\} \text{2nd order mixed partial derivative}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = z_{yy} = f_{yy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \qquad \left[\because z_{xy} = z_{yx}\right.$$

$$\frac{\partial^3 z}{\partial x^3} = z_{xxx}$$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = z_{yxx} \qquad \left[\frac{\partial^3 z}{\partial x^2 \partial y} = \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial y \partial x^2} \right.$$

partial derivative z, twice wrt x & once wrt y

$$\frac{\partial^3 z}{\partial x \partial y^2} = z_{yyx} \qquad \left[\frac{\partial^3 z}{\partial x \partial y^2} = \frac{\partial^3 z}{\partial y \partial x \partial y} = \frac{\partial^3 z}{\partial y^2 \partial x} \right.$$

partial derivative of z, once wrt x & twice wrt y

$$\frac{\partial^3 z}{\partial y^3} = z_{yyy}$$

Prove that $y v_y - x v_x = -y^2 v^3$ if $v = \left(1-2xy+y^2\right)^{-1/2}$.

Ans:- $v_y = \frac{\partial v}{\partial y} \qquad v_x = \frac{\partial v}{\partial x}$

$$v = (1-2xy+y^2)^{-1/2} \longrightarrow (1)$$

$$v = \frac{1}{(1-2xy+y^2)^{1/2}} \Longrightarrow \frac{1}{v^2} = 1-2xy+y^2 \longrightarrow (2)$$

$$\frac{\partial}{\partial x}\{\text{eqn } (2)\} \Longrightarrow -\frac{2}{v^3}\frac{\partial v}{\partial x} = 0 - 2y(1) + 0$$

$\left[\frac{\partial v^{-2}}{\partial x} = -2v^{-2-1}\frac{\partial v}{\partial x} \right.$

y is treated as constant

$$\frac{\partial v}{\partial x} = v^3 y \longrightarrow (3)$$

$$\frac{\partial}{\partial y}\{\text{eqn } (2)\} \Longrightarrow -\frac{2}{v^3}\frac{\partial v}{\partial y} = -2x + 2y = -2(x-y)$$

$$\frac{\partial v}{\partial y} = v^3(x-y) \longrightarrow (4)$$

[x is treated as constant

$$\{y \times \text{eqn } (4)\} - \{x \times \text{eqn } (3)\}$$

$$y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} = y\{v^3(x-y)\} - x\{v^3 y\}$$

$$y v_y - x v_x = -y^2 v^3$$

Show that $u_x + u_y = u$, if $u = \dfrac{e^{x+y}}{e^x + e^y}$.

Ans:- $u = \dfrac{e^{x+y}}{e^x + e^y}$

take $\log_e$ on both sides

$$\log u = \log\{e^{x+y}\} - \log(e^x + e^y)$$

$$\log(u) = x + y - \log(e^x + e^y) \longrightarrow ① \quad [\because \log_e e = 1$$

$$\frac{\partial}{\partial x}\{\text{eqn } ①\} \Longrightarrow \frac{1}{u}\frac{\partial u}{\partial x} = 1 + 0 - \left\{\frac{1}{e^x + e^y} \times (e^x + 0)\right\} \quad [\text{treat } y \text{ as constant}$$

$$\frac{\partial u}{\partial x} = u\left[1 - \frac{e^x}{e^x + e^y}\right] \longrightarrow ②$$

$$\frac{\partial}{\partial y}\{\text{eqn } ①\} \Longrightarrow \frac{1}{u}\frac{\partial u}{\partial y} = 0 + 1 - \left\{\frac{1}{e^x + e^y}(0 + e^y)\right\} \quad [\text{treat } x \text{ as constant}$$

$$\frac{\partial u}{\partial y} = u\left[1 - \frac{e^y}{e^x + e^y}\right] \longrightarrow ③$$

$$② + ③ \Longrightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u\left[1 - \frac{e^x}{e^x + e^y} + 1 - \frac{e^y}{e^x + e^y}\right]$$

$$= u\left[2 - \frac{e^x + e^y}{e^x + e^y}\right]$$

$$= u[2 - 1] = u$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

HW If $w = x^2y + y^2z + z^2x$, prove that $w_x + w_y + w_z = (x + y + z)^2$.

If $z = e^{ax+by} f(ax-by)$ prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Ans:- $z = e^{ax+by} f(ax-by)$

Let $ax - by = u \Rightarrow z = e^{ax+by} f(u) \longrightarrow (1)$

$\frac{\partial}{\partial x}\{\text{eqn } (1)\} \Longrightarrow$

[apply product rule y treated as constant

$$\frac{\partial z}{\partial x} = \{e^{ax+by}(a+0)\} f(u) + e^{ax+by} \frac{df}{du}\frac{\partial u}{\partial x}$$

$\left[\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}\right.$

$f = f(u)$

ordinary or total

partial

$$\frac{\partial z}{\partial x} = a e^{ax+by} f(u) + e^{ax+by}\frac{df}{du}(a-0)$$

$$\frac{\partial z}{\partial x} = az + a e^{ax+by}\frac{df}{du} \longrightarrow (2)$$

$\frac{\partial}{\partial y}\{\text{eqn } (1)\} \Longrightarrow$

apply product rule x is treated as constant

$$\frac{\partial z}{\partial y} = \{e^{ax+by}(0+b)\} f(u) + e^{ax+by}\frac{df}{du}(0-b)$$

$\left[\frac{\partial f}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y}\right.$

$$= b e^{ax+by} f(u) - b e^{ax+by}\frac{df}{du}$$

$$\frac{\partial z}{\partial y} = bz - b e^{ax+by}\frac{df}{du} \longrightarrow (3)$$

$b \times \text{eqn } (2) + a \times \text{eqn } (3)$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = b\left\{az + a e^{ax+by}\frac{df}{du}\right\} + a\left\{bz - b e^{ax+by}\frac{df}{du}\right\}$$

$$= 2abz$$

$u=\log_e\left(x^3+y^3-x^2y-xy^2\right)$, then show that $u_{xx}+2u_{xy}+u_{yy}=-\frac{4}{(x+y)^2}$.

Ans:- $u_{xx}=\frac{\partial^2 u}{\partial x^2}$ $\quad u_{xy}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$ $\quad u_{yy}=\frac{\partial^2 u}{\partial y^2}$

$$u=\log(x^3+y^3-x^2y-xy^2)$$

$$u=\log(x^3-x^2y+y^3-xy^2)$$

$$u=\log\{x^2(x-y)+y^2(y-x)\}=\log\{(x-y)(x^2-y^2)\}$$

$$u=\log\{(x-y)(x+y)(x-y)\}=\log\{(x-y)^2(x+y)\}$$

$$u=2\log(x-y)+\log(x+y)\longrightarrow ①$$

$$\frac{\partial u}{\partial x}=2\frac{1}{x-y}(1-0)+\frac{1}{x+y}(1+0)$$ [y is treated as constant

$$\frac{\partial u}{\partial x}=\frac{2}{x-y}+\frac{1}{x+y}\longrightarrow ②$$

$$\frac{\partial u}{\partial y}=2\frac{1}{x-y}(0-1)+\frac{1}{x+y}(0+1)$$ [x is treated as constant

$$\frac{\partial u}{\partial y}=\frac{-2}{x-y}+\frac{1}{x+y}\longrightarrow ③$$

$$\frac{\partial}{\partial x}(\text{eqn } ②)\Rightarrow\frac{\partial^2 u}{\partial x^2}=2\left\{\frac{-1}{(x-y)^2}(1-0)\right\}+\frac{-1}{(x+y)^2}(1+0)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{-2}{(x-y)^2}-\frac{1}{(x+y)^2}\longrightarrow ④$$

$$\frac{\partial}{\partial y}\{\text{eqn } ②\}\Rightarrow\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$$

$$\frac{\partial^2 u}{\partial x\partial y}=2\left\{\frac{-1}{(x-y)^2}(0-1)\right\}+\frac{-1}{(x+y)^2}(0+1)$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{2}{(x-y)^2}-\frac{1}{(x+y)^2}\longrightarrow ⑤$$

$$\frac{\partial}{\partial y}\{\text{eqn } 3\}$$

$$\Longrightarrow \frac{\partial^2 u}{\partial y^2} = -2\left\{\frac{-1}{(x-y)^2}(0-1)\right\} + \frac{-1}{(x+y)^2}(0+1)$$

$$\frac{\partial u}{\partial y} = \frac{-2}{x-y} + \frac{1}{x+y} \longrightarrow 3$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} \longrightarrow 6$$

$$u_{xx} + 2u_{xy} + u_{yy} = (\quad)$$

eqn 4 + 2 × eqn 5 + eqn 6

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2} + 2\left(\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}\right)$$

$$+ \left(\frac{-2}{(x-y)^2} - \frac{1}{(x+y)^2}\right)$$

$$= \frac{-2+4-2}{(x-y)^2} - \frac{(1+2+1)}{(x+y)^2} = \frac{-4}{(x+y)^2}$$

Qn) If $u = \log_e\left[x^3 + y^3 + z^3 - 3xyz\right]$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

$u = f(x,y,z)$ Now x, y, z are independent [previous prob $z = g(x,y)$ independent

u dependent

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad 1$$

if $\frac{d}{dx} = D$ then $\frac{d^2y}{dx^2} = D^2 y$

Ans let $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = D_1$

$$D_1 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = v \longrightarrow 2$$

$$D_1^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = D_1[D_1 u]$$

[Similar to $D^2 y$
$D^2 y = D[Dy]$ from 1

$$D_1^2 u = D_1 v = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)v = \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z} \longrightarrow (3)$$

[other wise $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+2\frac{\partial^2}{\partial x\partial y}+2\frac{\partial^2}{\partial x\partial z}+2\frac{\partial^2}{\partial y\partial z}$

which is tedious

$$\boxed{u = \log(x^3+y^3+z^3-3xyz)}$$

Changing $x \to y \to z \to x$ No change in u

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) \longrightarrow (3a)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\{3x^2+0+0-3yz\}$$

[treat y & z as constants

$$\frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz} \longrightarrow (4)$$

$$\frac{\partial u}{\partial y} = \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz} \longrightarrow (5)$$

[treat x & z as constant

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz} \longrightarrow (6)$$

[treat x & y as constants

$$(4)+(5)+(6) \Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(x^2-yz+y^2-zx+z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$D_1 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{3}{x+y+z} = v \longrightarrow (2)$$

from (3) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = D_1^2 u = D_1 v = \frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}$

$$\frac{\partial v}{\partial x} = \frac{-3}{(x+y+z)^2}(1+0+0)$$

[$\because$ treat y & z as constants

$$\frac{\partial v}{\partial y} = -\frac{3}{(x+y+z)^2} \qquad \frac{\partial v}{\partial z} = \frac{-3}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

x, y = independent variable
u = dependent variable (depends on x & y)

If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, then show that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Ans:- $$\frac{\partial u}{\partial x} = \left[2x\tan^{-1}\left(\frac{y}{x}\right) + x^2\frac{1}{1+\left(\frac{y}{x}\right)^2}\left\{y\left(-\frac{1}{x^2}\right)\right\}\right] - \left\{0\tan^{-1}\left(\frac{x}{y}\right) + y^2\left[\left\{\frac{1}{1+\left(\frac{x}{y}\right)^2}\right\}\frac{1}{y}\right]\right\}$$

[differentiate u w.r.t x keeping others const (i.e. y)]

similar to $\frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{x}\right)\right]$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{y^2+x^2}$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y(x^2+y^2)}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = x^2\left\{\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\right\} - \left[2y\tan^{-1}\left(\frac{x}{y}\right) + y^2\frac{1}{1+\left(\frac{x}{y}\right)^2}\left\{x\left(-\frac{1}{y^2}\right)\right\}\right]$$

differentiate u w.r.t y keeping x as constant

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2 x}{y^2+x^2}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial y} = x - 2y\tan^{-1}\left(\frac{x}{y}\right) \longrightarrow (2)$$

$$\frac{\partial}{\partial y}\{\text{eqn }(1)\} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 2x\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\left\{\frac{1}{x}\right\}\right] - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1 = \frac{2x^2-(x^2+y^2)}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{x^2-y^2}{x^2+y^2} \longrightarrow (3)$$

$$\frac{\partial}{\partial x}\{\text{eqn }(2)\} \Rightarrow \frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial y}\right\} = 1 - 2y\left[\frac{1}{1+\left(\frac{x}{y}\right)^2}\left\{\frac{1}{y}\right\}\right]$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - 2\frac{y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} \longrightarrow (4)$$

from (3) and (4) $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

Given $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$, Prove that $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.

Ans:- $u = e^{r\cos(\theta)}\cos(r\sin\theta) \longrightarrow (1)$

$$\frac{\partial u}{\partial r} = \left[e^{r\cos\theta}\{\cos(\theta)\}\right]\cos(r\sin\theta) + e^{r\cos(\theta)}\left[-\sin(r\sin\theta)\{\sin\theta\}\right]$$

Product rule should be applied

be partial differential wrt r treating θ as constant

$\frac{d}{dr}\{e^{ar}\}$ $\quad$ $\frac{d}{dr}\{\cos(br)\}$

$$\frac{\partial u}{\partial r} = e^{r\cos(\theta)}\{\cos(\theta)\cos(r\sin\theta) - \sin(\theta)\sin(r\sin\theta)\}$$

$$\frac{\partial u}{\partial r} = e^{r\cos\theta}\cos(\theta + r\sin\theta) \longrightarrow (2)$$

$$\frac{\partial u}{\partial \theta} = \left[e^{r\cos\theta}\{(r)(-\sin\theta)\}\right]\cos(r\sin\theta) + e^{r\cos(\theta)}\left[-\sin(r\sin\theta)\{r\cos\theta\}\right]$$

Differentiate partially wrt θ treating r as constant

$$\frac{\partial u}{\partial \theta} = -r\,e^{r\cos(\theta)}\{\sin(\theta)\cos(r\sin\theta) + \cos\theta\sin(r\sin(\theta))\}$$

$$= -re^{r\cos\theta}\sin\{\theta + r\sin(\theta)\} \longrightarrow (3)$$

$$\frac{\partial}{\partial\theta}\{e^{a\cos\theta}\} = e^{a\cos\theta}\{a\{-\sin\theta\}\} \qquad \frac{d\,ku}{dx} = k\frac{du}{dx}$$

$$v = e^{r\cos(\theta)}\sin(r\sin\theta)$$

$$\frac{\partial v}{\partial r} = \left[e^{r\cos(\theta)}\{\cos\theta\}\right]\sin(r\sin\theta) + e^{r\cos(\theta)}\left[\cos(r\sin\theta)\{\sin(\theta)\}\right]$$

$\frac{d}{dr}\{\sin(ar)\}$ $\quad a = \sin\theta$

$$= e^{r\cos(\theta)}\{\sin(r\sin\theta)\cos\theta + \cos(r\sin\theta)\sin\theta\}$$

$$\frac{\partial v}{\partial r} = e^{r\cos(\theta)}\sin(r\sin(\theta) + \theta) \longrightarrow (4)$$

$$\frac{\partial v}{\partial \theta} = \left[e^{r\cos(\theta)}\{r(-\sin\theta)\}\right]\sin(r\sin\theta) + e^{r\cos(\theta)}\left[\cos(r\sin\theta)\{r\cos\theta\}\right]$$

$$= re^{r\cos(\theta)}\left[\cos(r\sin\theta)\cos(\theta) - \sin(r\sin\theta)\sin(\theta)\right]$$

$$= re^{r\cos(\theta)}\cos\{r\sin(\theta)+\theta\} \longrightarrow (5)$$

From eqn (3) $\quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = e^{r\cos(\theta)}\sin(r\sin(\theta)+\theta)$

Using (4) $\quad \boxed{-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}}$

From eqn (5) $\quad \frac{1}{r}\frac{\partial v}{\partial \theta} = e^{r\cos(\theta)}\cos(r\sin(\theta)+\theta)$

Using eqn (2) $\Longrightarrow \boxed{\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}}$

If $(x+y)z = x^2+y^2$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$.

Ans $\quad z = g(x,y) \qquad z = \frac{x^2+y^2}{x+y} \longrightarrow$ requires quotient rule

$(x+y)z = x^2+y^2 \longrightarrow (1) \qquad \longrightarrow$ product rule

$$\frac{\partial}{\partial x}\{\text{eqn } (1)\} \Longrightarrow (1+0)z + (x+y)\frac{\partial z}{\partial x} = 2x+0$$

$$\frac{\partial z}{\partial x} = \frac{2x-z}{x+y} \longrightarrow (2)$$

$$\frac{\partial}{\partial y}\{\text{eqn } (1)\} \Rightarrow (0+1)z + (x+y)\frac{\partial z}{\partial y} = 0+2y$$

$$\frac{\partial z}{\partial y} = \frac{2y-z}{x+y} \longrightarrow (3)$$

Subtract (3) from (2)

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2x-z}{x+y} - \left(\frac{2y-z}{x+y}\right) = \frac{2x-z-2y+z}{x+y}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{2(x-y)}{x+y}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\frac{(x-y)^2}{(x+y)^2} \longrightarrow (4)$$

Consider $1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \left(\frac{2x-z}{x+y}\right) - \left(\frac{2y-z}{x+y}\right)$$

$$= \frac{(x+y) - 2x + z - 2y + z}{x+y}$$

$$= \frac{2z - x - y}{x+y} \longrightarrow (5)$$

From (1) $z = \frac{x^2+y^2}{x+y}$. [Put this in (5)

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2\left(\frac{x^2+y^2}{x+y}\right) - (x+y)}{x+y}$$

$$= \frac{\frac{2(x^2+y^2) - (x+y)^2}{(x+y)}}{x+y} = \frac{2x^2+2y^2-(x^2+2xy+y^2)}{(x+y)^2}$$

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2+y^2-2xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2} \longrightarrow (6)$$

Eqn (4) is $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\frac{(x-y)^2}{(x+y)^2} \longrightarrow (4)$

(6) in (4) $\Rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left\{1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right\}$

Prove LHS = RHS

(simplify) $\circleddash$ (simplify) → Instead of proving already assumed } Wrong procedure

Simplify = (simplify

LHS → simplify (eqn N) RHS separately → simplify → eqn M

If $v = \frac{1}{\sqrt{t}} e^{-x^2/4a^2 t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.

Ans

$$\log v = \log\left(t^{-1/2}\right) + \log e^{-(x^2/4a^2 t)}$$

$$\log v = -\frac{1}{2}\log(t) - \frac{x^2}{4a^2 t} \longrightarrow (1) \quad \left[\because \log_e e = 1\right.$$

$$\underbrace{\frac{\partial}{\partial x}\{(1)\}}_{\text{treat 't' as constant}} \Longrightarrow \frac{1}{v}\frac{\partial v}{\partial x} = 0 - \frac{2x}{4a^2 t}$$

$$\frac{\partial v}{\partial x} = -\frac{vx}{2a^2 t} \longrightarrow (2)$$

$$\frac{\partial}{\partial x}\{\text{eqn } (2)\} \Longrightarrow \frac{\partial^2 v}{\partial x^2} = -\frac{1}{2a^2 t}\underbrace{\frac{\partial}{\partial x}\{vx\}}_{\text{product rule}}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{2a^2 t}\left\{\frac{\partial v}{\partial x}x + v\right\} = -\frac{1}{2a^2 t}\left\{\left(\frac{-vx}{2a^2 t}\right)x + v\right\}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{v}{2a^2 t}\left\{\frac{x^2}{2a^2 t} - 1\right\} \longrightarrow (3)$$

$$\underbrace{\frac{\partial}{\partial t}\{\text{eqn } (1)\}}_{\text{treat } x \text{ as constant}} \Longrightarrow \frac{1}{v}\frac{\partial v}{\partial t} = -\frac{1}{2}\frac{1}{t} - \frac{x^2}{4a^2}\left(-\frac{1}{t^2}\right)$$

$$\frac{\partial v}{\partial t} = v\left\{\frac{x^2}{4a^2 t^2} - \frac{1}{2t}\right\} \longrightarrow (4)$$

$a^2 \times$ eqn ③ $\Longrightarrow$ $a^2 \dfrac{\partial^2 v}{\partial x^2} = a^2 \dfrac{v}{2a^2 t}\left\{\dfrac{x^2}{2a^2 t} - 1\right\}$

$$a^2 \frac{\partial^2 v}{\partial x^2} = v\left\{\frac{x^2}{4a^2t^2} - \frac{1}{2t}\right\} \longrightarrow ⑤$$

from ④ and ⑤ $\Longrightarrow$ $a^2 \dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}$ $\longrightarrow$ 1-D heat or diffusion equation

Q.) If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\dfrac{1}{r^2}\left[\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right)\right] = \dfrac{\partial \theta}{\partial t}$. $\rightarrow$ step 4 or

$\hookrightarrow$ preference is take log

$\rightarrow$ 1st step

$\rightarrow$ 2nd step

$\rightarrow$ step 3

$$\log \theta = n \log t - \frac{r^2}{4t} \longrightarrow ①$$

$$\frac{\partial}{\partial t}\{\text{eqn } 1\} \Longrightarrow \frac{1}{\theta}\frac{\partial \theta}{\partial t} = \frac{n}{t} - \frac{r^2}{4}\left(\frac{-1}{t^2}\right)$$

$$\frac{\partial \theta}{\partial t} = \frac{\theta}{t}\left\{n + \frac{r^2}{4t}\right\} \longrightarrow ②$$

$$\frac{\partial}{\partial r}\{\text{eqn } ①\} \Longrightarrow \frac{1}{\theta}\frac{\partial \theta}{\partial r} = 0 - \frac{1}{4t}(2r) = -\frac{r}{2t}$$

$$\frac{\partial \theta}{\partial r} = \frac{-\theta r}{2t}$$

$$r^2 \frac{\partial \theta}{\partial r} = -\frac{\theta r^3}{2t} \longrightarrow ③$$

$$\frac{\partial}{\partial r}\{\text{eqn } ③\} \Longrightarrow \frac{\partial}{\partial r}\left\{r^2 \frac{\partial \theta}{\partial r}\right\} = -\frac{1}{2t}\frac{\partial}{\partial r}\left\{\theta r^3\right\} \quad \left[\because \theta = f(r,t)\right)$$

$$\frac{\partial}{\partial r}\left\{r^2 \frac{\partial \theta}{\partial r}\right\} = -\frac{1}{2t}\left\{\frac{\partial \theta}{\partial r} r^3 + \theta 3r^2\right\}$$

$$\frac{\partial}{\partial r}\left\{r^2 \frac{\partial \theta}{\partial r}\right\} = -\frac{1}{2t}\left\{\left(\frac{-\theta r}{2t}\right) r^3 + 3\theta r^2\right\}$$

$$\frac{\partial}{\partial r}\left\{r^2 \frac{\partial \theta}{\partial r}\right\} = \frac{\theta r^2}{t}\left\{\frac{r^2}{4t} - \frac{3}{2}\right\}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\theta}{\partial r}\right\} = \frac{\theta}{t}\left\{\frac{r^2}{4t}-\frac{3}{2}\right\} \longrightarrow (4)$$

Given $\quad \frac{1}{r}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\theta}{\partial r}\right\} = \frac{\partial\theta}{\partial t}$

$\therefore$ from eqns. (2) and (4)

$$\frac{\theta}{t}\left\{\frac{r^2}{4t}-\frac{3}{2}\right\} = \frac{\theta}{t}\left\{n+\frac{r^2}{4t}\right\}$$

$$\Longrightarrow \quad n = -\frac{3}{2}$$

HW Find the value of n so that the equation $v = r^n\left(3\cos^2\theta-1\right)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0.$$

Hint: $\left[\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial v}{\partial\theta}\right\} \div \sin\theta\right]$ then convert $\sin^2\theta = 1-\cos^2\theta$

$$n(n+1) - (\quad) = 0 \Longrightarrow n = n_1, n_2.$$

Qn) If $x^x y^y z^z = c$, show that $z_{xy} = -\left[x\log_e(ex)\right]^{-1}$, when $x=y=z$.

Ans:- $\quad f(x,y,z)=c \quad$ z depends on x and y but we can't express it as $z=g(x,y)$ and hence it is called implicit functions.

$$x^x y^y z^z = c$$

$$\log(x^x)+\log(y^y)+\log(z^z) = \log c$$

$$x\log(x)+y\log(y)+z\log(z) = \log c \longrightarrow (1)$$

$z_{xy} = \frac{\partial}{\partial y}(z_x) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ [But $z_{xy} = z_{yx}$

$\frac{\partial}{\partial x}\{\text{eqn } 1\}$ {product rule and chain rule in 3rd term

Remember z depends on x & y

$$\left\{\log x + x\frac{1}{x}\right\} + 0 + \left\{\frac{\partial z}{\partial x}\log(z) + z\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\right\} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{\{\log(x)+1\}}{\{\log(z)+1\}} \longrightarrow 2$$

$$\frac{\partial}{\partial y}\{\text{eqn } 2\} \Longrightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -\{\log(x)+1\}\left[\frac{-1}{\{\log(z)+1\}^2}\frac{1}{z}\frac{\partial z}{\partial y}\right]$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\log(x)+1}{z\{\log(z)+1\}^2}\frac{\partial z}{\partial y}$$

Cyclic change in x, y, z does not alter the function

$\therefore$ by observing 2 $\quad \frac{\partial z}{\partial y} = -\frac{\{\log(y)+1\}}{\{\log(z)+1\}}$

$$z_{xy} = \frac{\{\log(x)+1\}}{z\{\log(z)+1\}^2}\left[-\frac{\{\log(y)+1\}}{\log(z)+1}\right]$$

$$z_{xy} = -\frac{\{\log(x)+1\}\{\log(y)+1\}}{z\{\log(z)+1\}^3} \longrightarrow 3$$

At $x=y=z$ [$\because y=x$ & $z=x$

In 2-D $x=y \longrightarrow$ st line 45° through $(0,0)$

In 3-D $x=y=z \longrightarrow$ line through $(0,0,0)$ makes equal angle with x,y,z-axis

On the line $x=y=z$ [$\because$ $y=x$ & $z=x$ in RHS only

$$z_{xy}\Big|_{x=y=z} = \frac{-\{\log(x)+1\}^2}{x\{\log(x)+1\}^3} = \frac{-1}{x\{\log(x)+1\}}$$

$$= -\left[x\{\log(x)+\underbrace{1}_{\log_e e}\}\right]^{-1}$$

$$= -\left[x\{\log(x)+\log e\}\right]^{-1}$$

$$= -\left[x\log(ex)\right]^{-1}$$

If $w=r^m$, prove that $\underbrace{w_{xx}+w_{yy}+w_{zz}}_{\nabla^2 w}=m(m+1)r^{m-2}$ where $r^2=x^2+y^2+z^2$.

Ans:- $\underbrace{\nabla^2}_{\text{Laplacian operator}} w$

$w=r^m \qquad r^2=x^2+y^2+z^2$ → My preference

[$\because$ $w=(x^2+y^2+z^2)^{m/2}$ This is another way

Changing x,y,z in cyclic order does not change the function, $\therefore$ derivatives can be written by observation

$$\frac{\partial w}{\partial x} = m r^{m-1}\frac{\partial r}{\partial x} \longrightarrow (1)$$

$$r^2=x^2+y^2+z^2 \Longrightarrow 2r\frac{\partial r}{\partial x} = 2x$$

[$\because$ treat y & z as constants in $r=r(x,y,z)$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \longrightarrow (2)$$

Eqn (2) in eqn (1) $\Longrightarrow$ $\frac{\partial w}{\partial x} = w_x = m r^{m-1}\frac{x}{r} = m r^{m-2}x$

$$w_x = m r^{m-2}x \longrightarrow (3)$$

$$\Longrightarrow \quad W_y = m r^{m-2} y \qquad \text{and} \qquad W_z = m r^{m-2} z$$

$$\frac{\partial}{\partial x}\{\text{eqn } 3\} \Longrightarrow \frac{\partial^2 W}{\partial x^2} = \frac{\partial}{\partial x}(W_x) = m\frac{\partial}{\partial x}(r^{m-2}x)$$

[Product rule to be applied as $r = r(x, y, z)$)

$$W_{xx} = m\left[\left\{(m-2)r^{m-3}\frac{\partial r}{\partial x}\right\}x + r^{m-2}\right]$$

$$W_{xx} = m\left[\left\{(m-2)r^{m-3}\frac{x}{r}\right\}x + r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] \longrightarrow 4$$

Observing eqn 4

$$W_{yy} = m\left[(m-2)r^{m-4}y^2 + r^{m-2}\right] \longrightarrow 5$$

$$W_{zz} = m\left[(m-2)r^{m-4}z^2 + r^{m-2}\right] \longrightarrow 6$$

4 + 5 + 6 $\Longrightarrow$

$$W_{xx} + W_{yy} + W_{zz} = m\left[(m-2)r^{m-4}\underbrace{(x^2+y^2+z^2)}_{= r^2} + 3r^{m-2}\right]$$

$$\nabla^2 W = m\left[(m-2)\underbrace{r^{m-4}\, r^2}_{= r^{m-4+2} = r^{m-2}} + 3r^{m-2}\right]$$

$$= m r^{m-2}(m-2+3) = m(m+1)r^{m-2}$$

HW Verify whether $v = e^{3x+4y}\cos(5z)$ satisfies the Laplace equation.

[Hint: Laplace equation: $\nabla^2 \psi = 0$

To check whether $\nabla^2 v = 0$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$f: v = v(x,y,z)$

Find $W_x, W_y, W_z, W_{xx}, W_{yy}, W_{zz}$ independently as function changes with cyclic change in x, y, z

$u = x^y$

$$\frac{\partial u}{\partial x} = y x^{y-1} \qquad \frac{\partial u}{\partial y} = x^y \log(x)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(y x^{y-1}\right) = x^{y-1} + y\, x^{y-1}\log(x) \longrightarrow (1)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^y \log x\right) = \left(y x^{y-1}\right)\log(x) + \left(x^y\right)\frac{1}{x}$$

$$= y x^{y-1}\log(x) + x^{y-1} \longrightarrow (2)$$

From (1) and (2) $\quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

from (1) from (2)

Total derivatives.

* Total differential, (gives total change in the function

$u = u(x,y)$ then total differential is

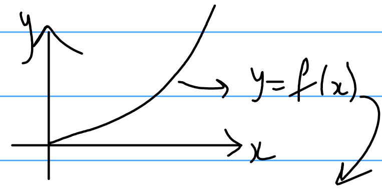
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

* Total derivative of $u(x,y)$ wrt 't'

[∵ rate of change of u wrt 'a' curve in the xy plane]

[∵ $u = u(x,y)$ $x = x(t)$ $y = y(t)$ [∵ a

Total derivative is $\frac{du}{dt}$



or

$x = x(t)$ $y = f(t)$

Eg: $x = t$ $y = \frac{t^2}{4a}$

Now $du = \frac{\partial u}{\partial x}(dx) + \frac{\partial u}{\partial y}(dy)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

Similary if $y = f(x)$ or $f(x,y) = c$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

OR

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\frac{dx}{dy} + \frac{\partial u}{\partial y}$$

Find the differential of the following functions:

a) $f(x, y) = x\cos y - y\cos x$ HW

b) $f(x, y, z) = e^{xyz}$ $\longrightarrow$ class

Ans:- b) total differential, $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$$f(x,y,z) = e^{xyz}$$

$$\frac{\partial f}{\partial x} = e^{xyz}\, yz \quad \left[\because\ y,z \text{ are treated as constants for } \frac{\partial f}{\partial x} \text{ of } f(x,y,z) \text{ i.e } f = e^{ax} \text{ for } \frac{\partial f}{\partial x}\right.$$

Similarly $\dfrac{\partial f}{\partial y} = e^{xyz}\, xz \quad \left[\because\ f = e^{ay} \text{ for } \dfrac{\partial f}{\partial y}\right.$

$$\frac{\partial f}{\partial z} = e^{xyz}\, xy \quad \left[\because\ f = e^{az} \text{ for } \frac{\partial f}{\partial z}\right.$$

$$df = e^{xyz}\, yz\, dx + e^{xyz}\, xz\, dy + e^{xyz}\, xy\, dz$$

Eg) $A = xy \qquad \dfrac{dA}{dt} = \dfrac{dx}{dt}\, y + x\, \dfrac{dy}{dt} \longrightarrow$ +2 class

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy \Longrightarrow \frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt}$$

$$\frac{dA}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$ We get the same

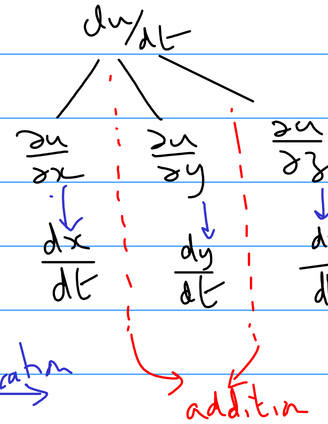
This is simple eg. When A is little complicated then we can get answer in elegant

Find $\dfrac{du}{dt}$ and also verify the result by direct substitution if $u = x^2 + y^2 + z^2,\ x = e^{2t},\ y = e^{2t}\cos 3t$ and $z = e^{2t}\sin 3t$

$\to u = f(x,y,z)$

Ans) $u \longrightarrow (x,y,z) \longrightarrow t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$



$$\frac{\partial u}{\partial x} = 2x + 0 + 0 = 2x$$

$\hookrightarrow$ diff u wrt x treating y, z as constant

$$\frac{\partial u}{\partial y} = 0+2y+0 = 2y \qquad\qquad \frac{\partial u}{\partial z} = 0+0+2z$$

$\Rightarrow x, z$ are treated as constants $\qquad \Rightarrow x, y$ are treated as constants

$$x = e^{2t} \Longrightarrow \frac{dx}{dt} = 2e^{2t}$$

$$y = e^{2t}\cos(3t) \Longrightarrow \frac{dy}{dt} = \{2e^{2t}\}\cos(3t) + e^{2t}\{-\sin(3t).3\}$$

$$\frac{dy}{dt} = e^{2t}\{2\cos(3t) - 3\sin(3t)\}$$

$$z = e^{2t}\sin(3t) \Rightarrow \frac{dz}{dt} = \{2e^{2t}\}\sin(3t) + e^{2t}\{\cos(3t).3\}$$

$$\frac{dz}{dt} = e^{2t}\{2\sin(3t) + 3\cos(3t)\}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= 2x(2e^{2t}) + 2y\left[e^{2t}\{2\cos(3t) - 3\sin(3t)\}\right] + 2z\left[e^{2t}\{2\sin(3t)+3\cos(3t)\}\right]$$

$$= 2e^{2t}\left[e^{2t}(2) + e^{2t}\cos(3t)\{2\cos(3t) - 3\sin(3t)\} + e^{2t}\sin(3t)\{2\sin(3t)+3\cos(3t)\}\right]$$

$$= 2e^{4t}\left[2 + 2\cos^2(3t) - 3\cos(3t)\sin(3t) + 2\sin^2(3t) + 3\sin(3t)\cos(3t)\right]$$

$$= 2e^{4t}\left[2 + 2\{\cos^2(3t) + \sin^2(3t)\}\right]$$

$$\frac{du}{dt} = 2e^{4t}(2+2) = 8e^{4t} \longrightarrow (1)$$

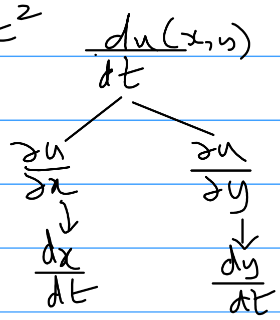
$$u = x^2+y^2+z^2 = (e^{2t})^2 + \{e^{2t}\cos(3t)\}^2 + \{e^{2t}\sin(3t)\}^2$$

$$= e^{4t} + e^{4t}\{\cos^2(3t) + \sin^2(3t)\} = 2e^{4t}$$

$$\frac{du}{dt} = 2(4e^{4t}) = 8e^{4t} \longrightarrow (2)$$

From (1) and (2) results are verified

If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = 3\left(1-t^2\right)^{-\frac{1}{2}}$. $= \frac{3}{\sqrt{1-t^2}}$



Ans:- $\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}}(1-0) = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}}(0-1) = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 12t^2$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}(3) + \frac{-1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}} = \frac{3(1-4t^2)}{\sqrt{1-(3t-4t^3)^2}}$$

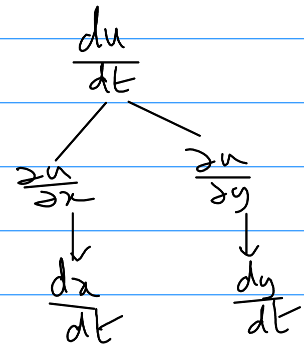
$$= \frac{3(1-4t^2)}{\sqrt{1-(9t^2-24t^4+16t^6)}} = \frac{3(1-4t^2)}{\sqrt{1-9t^2+24t^4-16t^6}}$$

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}} = \frac{3}{\sqrt{1-t^2}}$$

$$\begin{array}{r|l} & 1-8t^2+16t^4 \\ \hline (1-t^2) & 1-9t^2+24t^4-16t^6 \\ & 1-t^2 \\ \hline & -8t^2+24t^4-16t^6 \\ & -8t^2+8t^4 \\ \hline & 16t^4-16t^6 \\ & 16t^4-16t^6 \\ \hline & 0 \quad - \quad 0 \end{array}$$

$u = x^y$, when $y = \tan^{-1}t$, $x = \sin t$. Find $\frac{du}{dt}$



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$\frac{\partial u}{\partial x} = y x^{y-1}$ (Similar to $\frac{d}{dx}(x^n)$)

$\Longrightarrow$ y is treated as constant

$$\frac{\partial u}{\partial y} = x^y \log(x) \qquad \left[\text{Similar to } \frac{d}{dy}(a^y) = a^y \log(a)\right.$$

$\Longrightarrow x$ is treated as constant

$$\frac{dx}{dt} = \cos(t) \qquad \frac{dy}{dt} = \frac{1}{1+t^2}$$

$$\frac{du}{dt} = y x^{y-1} \cos(t) + x^y \log(x) \frac{1}{1+t^2}$$

The altitude of a right circular cone is $15cm$ and is increasing at $0.2cm/s$. The radius of the base is $10cm$ and is decreasing at $0.3cm/s$. How fast is the volume changing?

Ans:- $V = \frac{1}{3}\pi r^2 h \Longrightarrow V = f(r, h)$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$\frac{dV}{dt} = \left\{\frac{1}{3}\pi(2r)h\right\}\frac{dr}{dt} + \left(\frac{1}{3}\pi r^2\right)\frac{dh}{dt} \longrightarrow (1)$$

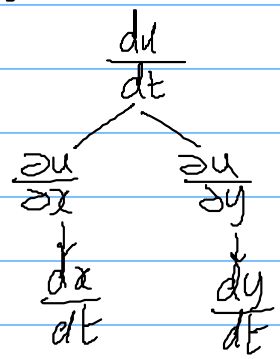
At $h = 15 \quad r = 10 \quad \frac{dr}{dt} = -0.3 \quad \frac{dh}{dt} = 0.2$

$$\frac{dV}{dt} = \frac{\pi}{3}\left\{2(10)(15)(-0.3) + (10)^2(0.2)\right\} = -\frac{70\pi}{3}$$

In order that the function $u = 2xy - 3x^2y$ remains constant, what should be the rate of change of y w.r.t. t, given x increases at the rate of 2cm/sec at the instant when x = 3cm and y = 1cm.

Ans:- $u = 2xy - 3x^2y$

At $x = 3 \quad y = 1 \quad \frac{dx}{dt} = 2\,cms^{-1} \quad \frac{du}{dt} = 0$



$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = (2y - 6xy)\frac{dx}{dt} + (2x - 3x^2)\frac{dy}{dt}$$

At $x = 3 \quad y = 1$

$$0 = \{2(1) - 6(3)(1)\}(2) + \{2(3) - 3(3)^2\}\frac{dy}{dt}$$

$\frac{dy}{dt} = -\frac{32}{21}\,cms^{-1}$ [∴ y decreases at the rate of $32/21\,cms^{-1}$

Implicit functions $f(x,y) = c$. find $\frac{dy}{dx}$

Now $\boxed{\frac{df}{dx} = 0}$ if $f(x,y) = c$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\frac{df}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Eg:- $2x^3y^2 + 3xy^3 = c$ find $\frac{dy}{dx}$

Ans:- $f(x,y) = 2x^3 + 3xy^3$ then $\frac{dy}{dx} = \frac{-f_x}{f_y}$

$f_x = 6x^2y^2 + 3y^3 \qquad f_y = 4x^3y + 9xy^2$

$$\frac{dy}{dx} = -\frac{(6x^2y^2 + 3y^3)}{4x^3y + 9xy^2}$$

Total derivative $\frac{df}{dt}$ $\quad f \to (x,y) \quad x,y \to t$

$$\begin{tikzcd}[column sep=small] & \frac{df}{dt} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{\partial f}{\partial x} \arrow[d] & & \frac{\partial f}{\partial y} \arrow[d] \\ \frac{dx}{dt} & & \frac{dy}{dt} \end{tikzcd}$$

$$f \longrightarrow (x,y) \longrightarrow (r,s)$$

$\cancel{\frac{df}{dr}}$ $\quad \frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial s}$

$$\begin{tikzcd}[column sep=small] & \frac{\partial f}{\partial r} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{\partial f}{\partial x} \arrow[d] & & \frac{\partial f}{\partial y} \arrow[d] \\ \frac{\partial x}{\partial r} & & \frac{\partial y}{\partial r} \end{tikzcd}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

If $W = u^2 v$ and $u = e^{x^2-y^2}$, $v = \sin(xy^2)$ find $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$.

$$W \longrightarrow (u,v) \longrightarrow (x,y)$$

$\to$ 2 variable

$$\begin{tikzcd}[column sep=small] & \frac{\partial W}{\partial x} \arrow[dl, dash] \arrow[dr, dash] & \\ \frac{\partial W}{\partial u} \arrow[d] & & \frac{\partial W}{\partial v} \arrow[d] \\ \frac{\partial u}{\partial x} & & \frac{\partial v}{\partial x} \end{tikzcd}$$

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial x} \longrightarrow 2 \text{ terms}$$

$$\frac{\partial W}{\partial u} = 2uv \qquad \frac{\partial W}{\partial v} = u^2$$

$$\frac{\partial u}{\partial x} = \{e^{x^2-y^2}\}(2x-0) = 2x e^{x^2-y^2}$$

$$\frac{\partial v}{\partial x} = \cos(xy^2)\, y^2$$

$$\frac{\partial W}{\partial x} = (2uv)2xe^{x^2-y^2} + u^2y^2\cos(xy^2)$$

$$\frac{\partial W}{\partial y} = \frac{\partial W}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial W}{\partial v}\frac{\partial v}{\partial y}$$

$$u = e^{x^2-y^2} \implies \frac{\partial u}{\partial y} = e^{x^2-y^2}(0-2y) = -2ye^{x^2-y^2}$$

$$v = \sin(xy^2) \Longrightarrow \frac{\partial v}{\partial y} = \cos(xy^2)(x\,2y)$$

$$\frac{\partial w}{\partial y} = (2uv)\left\{-2y\,e^{x^2-y^2}\right\} + u^2\left\{2xy\cos(xy^2)\right\}$$

If $u = u\left[\dfrac{y-x}{xy}, \dfrac{z-x}{xz}\right]$, show that $x^2\dfrac{\partial u}{\partial x} + y^2\dfrac{\partial u}{\partial y} + z^2\dfrac{\partial u}{\partial z} = 0$.

$$u = f(x, y, z)$$

Let $r = \dfrac{y-x}{xy} = \dfrac{1}{x} - \dfrac{1}{y}$

$s = \dfrac{z-x}{xz} = \dfrac{1}{x} - \dfrac{1}{z}$

$u = f(x, y, z)$

$u = u(r, s)$ $\qquad r, s = g_i(x, y, z)$ $\qquad i = 1, 2$

$\partial u/\partial x$

$\dfrac{\partial u}{\partial r}$ $\quad \dfrac{\partial u}{\partial s}$

$\dfrac{\partial r}{\partial x}$ $\quad \dfrac{\partial s}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{-1}{x^2} \qquad \frac{\partial s}{\partial x} = \frac{-1}{x^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial s}\left(\frac{-1}{x^2}\right)$$

$$= -\frac{1}{x^2}\left\{\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right\} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

$$r = \frac{1}{x} - \frac{1}{y} \Longrightarrow \frac{\partial r}{\partial y} = 0 - \left\{-\frac{1}{y^2}\right\} = \frac{1}{y^2}$$

$$s = \frac{1}{x} - \frac{1}{z} \Longrightarrow \frac{\partial s}{\partial y} = 0$$ $\left[\because s \text{ is independent of } y\right.$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + 0 = \frac{1}{y^2}\frac{\partial u}{\partial r} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z}$$

$$\frac{\partial r}{\partial z} = 0 \qquad \left[\because\ r = \frac{1}{x} - \frac{1}{y}\right.$$

$$\frac{\partial s}{\partial z} = 0 - \left\{-\frac{1}{z^2}\right\} = \frac{1}{z^2} \qquad \left[\ s = \frac{1}{y} - \frac{1}{z}\right.$$

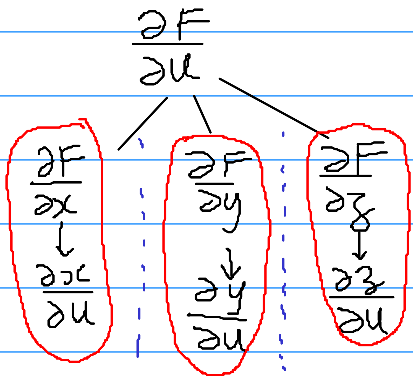
$$\frac{\partial u}{\partial z} = 0 + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right) = \frac{1}{z^2}\frac{\partial u}{\partial s} \longrightarrow (3)$$

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = x^2\left\{\left(-\frac{1}{x^2}\right)\left(\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right)\right\}$$
$$+ y^2\left\{\frac{1}{y^2}\frac{\partial u}{\partial r}\right\} + z^2\left\{\frac{1}{z^2}\frac{\partial u}{\partial s}\right\}$$
$$= -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} = 0$$

If $x = u + v + w, y = vw + wu + uv, z = uvw$ and F is a function of x, y, z show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

$$F \longrightarrow (x, y, z) \longrightarrow (u, v, w)$$



$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial x}{\partial u} = 1 + 0 + 0 \qquad \frac{\partial y}{\partial u} = 0 + w + v$$

$$\frac{\partial z}{\partial u} = vw$$

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x} + (w+v)\frac{\partial F}{\partial y} + vw\frac{\partial F}{\partial z} \longrightarrow (1)$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v}$$

$x = u+v+w$
$y = vw+wu+uv$
$z = uvw$

$$\frac{\partial x}{\partial v} = 0+1+0 \qquad \frac{\partial y}{\partial v} = w+0+u \qquad \frac{\partial z}{\partial v} = uw$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x} + (w+u)\frac{\partial F}{\partial y} + uw\frac{\partial F}{\partial z} \longrightarrow (2)$$

Similarly $\quad \dfrac{\partial F}{\partial w} = \dfrac{\partial F}{\partial x} + (u+v)\dfrac{\partial F}{\partial y} + uv\dfrac{\partial F}{\partial z} \longrightarrow (3)$

$u \times$ eqn (1) $+ v \times$ eqn (2) $+ w \times$ eqn (3) $\Longrightarrow$

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = u\left\{\frac{\partial F}{\partial x} + (v+w)\frac{\partial F}{\partial y} + vw\frac{\partial F}{\partial z}\right\}$$

$$+v\left\{\frac{\partial F}{\partial x} + (w+u)\frac{\partial F}{\partial y} + uw\frac{\partial F}{\partial z}\right\} + w\left\{\frac{\partial F}{\partial x} + (u+v)\frac{\partial F}{\partial y} + uv\frac{\partial F}{\partial z}\right\}$$

$$= (u+v+w)\frac{\partial F}{\partial x} + (\underline{uv}+\underline{uw}+\underline{vw}+\underline{vu}+\underline{uw}+\underline{vw})\frac{\partial F}{\partial y} + (uvw+uvw+uvw)\frac{\partial F}{\partial z}$$

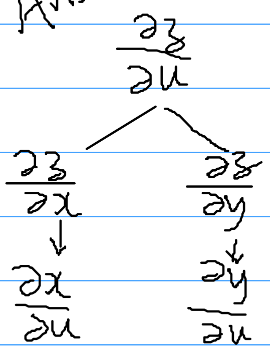
$$= x\frac{\partial F}{\partial x} + 2(uv+vw+wu)\frac{\partial F}{\partial y} + 3(uvw)\frac{\partial F}{\partial z}$$

$$= x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$

If z is a function of x and y and $x = e^u \cos v$, $y = e^u \sin v$.

Prove that (i) $x\dfrac{\partial z}{\partial v} + y\dfrac{\partial z}{\partial u} = e^{2u}\dfrac{\partial z}{\partial y}$ $\quad$ (ii) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2\right]$

Ans:- $\quad z \longrightarrow (x,y) \longrightarrow (u,v)$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial x}{\partial u} = e^u \cos v = x$$

$$\frac{\partial y}{\partial u} = e^u \sin(v) = y$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} x + \frac{\partial z}{\partial y} y \longrightarrow ①$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial x}{\partial v} = e^u\{-\sin(v)\} = -y$$

$$\frac{\partial y}{\partial v} = e^u\cos(v) = x$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-y) + \frac{\partial z}{\partial y} x = -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \longrightarrow ②$$

$x \times$ eqn ② $+ y \times$ eqn ①

$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = x\left\{-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right\} + y\left\{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right\}$$

$$= -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} + xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y}$$

$$= (x^2+y^2)\frac{\partial z}{\partial y}$$

$$x^2+y^2 = \{e^u\cos(v)\}^2 + \{e^u\sin(v)\}^2 = e^{2u}\{\cos^2 v + \sin^2 v\} = e^{2u}$$

$$x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$$

$\{\text{eqn } ①\}^2 + \{\text{eqn } ②\}^2$

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left\{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right\}^2 + \left\{-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right\}^2$$

$$= x^2\left(\frac{\partial z}{\partial x}\right)^2 + 2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + y^2\left(\frac{\partial z}{\partial y}\right)^2 + y^2\left(\frac{\partial z}{\partial x}\right)^2 - 2xy\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + x^2\left(\frac{\partial z}{\partial y}\right)^2$$

$$= (x^2+y^2)\left(\frac{\partial z}{\partial x}\right)^2 + (y^2+x^2)\left(\frac{\partial z}{\partial y}\right)^2$$

$$= e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left\{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right\}$$

Prove that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$ if z is a function of x and y and $x=e^{u}+e^{-v}$, $y=e^{-u}-e^{v}$.

$$z \longrightarrow (x,y) \longrightarrow (u,v)$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$=\frac{\partial z}{\partial x}(e^{u}+0)+\frac{\partial z}{\partial y}(-e^{-u}+0)=e^{u}\frac{\partial z}{\partial x}-e^{-u}\frac{\partial z}{\partial y} \longrightarrow (1)$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$=\frac{\partial z}{\partial x}(0-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})=-\left\{e^{-v}\frac{\partial z}{\partial x}+e^{v}\frac{\partial z}{\partial y}\right\} \longrightarrow (2)$$

Subtract (2) from (1)

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=e^{u}\frac{\partial z}{\partial x}-e^{-u}\frac{\partial z}{\partial y}+e^{-v}\frac{\partial z}{\partial x}+e^{v}\frac{\partial z}{\partial y}$$

$$=(e^{u}+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^{v})\frac{\partial z}{\partial y}$$

$$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$$

Test Portions

$$z \longrightarrow (x,y) \longrightarrow (u,v) \quad \left[\text{i.e } z=z(x,y)\right. \qquad x=x(u,v),\; y=y(u,v)$$

$$\frac{\partial z}{\partial u}$$

$$\frac{\partial z}{\partial x} \downarrow \frac{\partial x}{\partial u} \quad + \quad \frac{\partial z}{\partial y} \downarrow \frac{\partial y}{\partial u}$$

Multiply Multiply

$$\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

Replace u with v

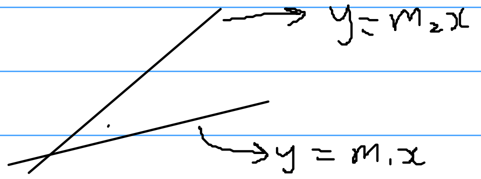
$$\frac{\partial z}{\partial y}$$

$$z \longrightarrow (u,v) \longrightarrow (x,y)$$

$$\frac{\partial z}{\partial x}$$

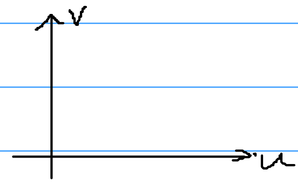
$$\frac{\partial z}{\partial u} \downarrow \frac{\partial u}{\partial x} \qquad \frac{\partial z}{\partial v} \downarrow \frac{\partial v}{\partial x}$$

Jacobians = determinant{Jacobian Matrix}



$u = y - m_1 x$

$v = y - m_2 x$



Jacobian of $u = u(x,y)$ $v = v(x,y)$ with respect to x, y is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$ is given by

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$ $\quad$ $J\left(\frac{x,y}{r,\theta}\right) = ?$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2(\theta) - \{r\sin^2\theta\} = r\{\cos^2\theta + \sin^2\theta\}$$

$$= r$$

Definition:- $u(x,y)$ and $v(x,y)$ are said to be functionally dependent if there exists a relation between u and v [i.e $u = f(v)$, $v = f(u)$ or $f(u,v) = c$

Theorem:- $u(x,y)$ and $v(x,y)$ are functionally dependent $\Longleftrightarrow$ $J\left(\frac{u,v}{x,y}\right) = 0$

Note:— The above definitions and the theorem are valid for n functions in n independent variable

Find the Jacobian of $u = e^x \sin(y)$ $v = x + \log\{\sin y\}$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} e^x \sin(y) & e^x \cos(y) \\ 1+0 & 0+\frac{1}{\sin(y)}\cos(y) \end{vmatrix}$$

$$= e^x \sin(y)\left\{\frac{\cos(y)}{\sin y}\right\} - e^x \cos(y)$$

$$= e^x \cos(y) - e^x \cos(y) = 0 \Longrightarrow$$ u, v are functionally dependent

$$\log(u) = \log e^x + \log(\sin(y)) = x + \log\{\sin(y)\}$$

$$\log(u) = v$$

Find the Jacobian of $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$.

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = 2x + 0 + 0$ $\quad u_y = 0 + 2y + 0$ $\quad u_z = 0 + 0 + 2z$

$v_x = y + 0 + z$ $\quad v_y = x + z + 0$ $\quad v_z = 0 + y + x$

$w_x = 1 + 0 + 0$ $\quad w_y = 0 + 1 + 0$ $\quad w_z = 0 + 0 + 1$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2x\{(x+z) - (x+y)\} - 2y\{(y+z) - (x+y)\} + 2z\{(y+z) - (x+z)\}$$

$$= 2\{x(z-y) - y(z-x) + z(y-x)\}$$

$$= 2\{xz - xy - yz + xy + yz - xz\} = 0 \Longrightarrow$$ u, v, w are functionally dependent

$$w^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = u + 2v$$

If $J\left(\frac{u,v}{x,y}\right) \neq 0$ then we can find

$x = x(u,v)$ and $y = y(u,v)$

$x = r\cos(\theta)$ $y = r\sin(\theta)$ $J\left(\frac{x,y}{r,\theta}\right) = r$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $J\left(\frac{r,\theta}{x,y}\right) = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

Property :- If $J\left(\frac{u,v}{x,y}\right) = J\,(\neq 0)$ and $J\left(\frac{x,y}{u,v}\right) = J'(\neq 0)$
then $JJ' = 1$

If $x = r\cos(\theta)$ $y = r\sin(\theta)$ prove that $JJ' = 1$

Ans let $J = J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin\theta \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$

$= r\cos^2\theta - (-r\sin^2\theta) = r$

$\because$ $r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2}$ $\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$J' = J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$

$r_x = \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x+0) = \frac{x}{\sqrt{x^2+y^2}}$ $r_y = \frac{y}{\sqrt{x^2+y^2}}$

$\theta_x = \frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left\{y\left(\frac{-1}{x^2}\right)\right\} = \frac{-y}{x^2+y^2}$

$\theta_y = \frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$

$$J' = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix}$$

$$J' = \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} - \frac{-y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$$JJ' = r\left(\frac{1}{r}\right) = 1$$

Find the Jacobian of
$u = x + 3y^2 - z^3, v = 4x^2yz, w = 2z^2 - xy$ at (1,-1,0).

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Ans: 20

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

At (1,-1,0) $J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$

$$= 1(-4) - (-6)(4) = -4 + 24 = 20$$

If $x = (a)\cosh\xi\cos\eta$, $y = (a)\sinh\xi\sin\eta$ Show that $\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)$

$\xi = X_i$ eta

$$J\left(\frac{x,y}{\xi,\eta}\right) = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix}$$

$$\frac{\partial x}{\partial \xi} = (a)\sinh(\xi)\cos(\eta) \qquad \frac{\partial x}{\partial \eta} = -(a)\cosh(\xi)\sin(\eta)$$

$$\frac{\partial y}{\partial \xi} = (a)\cosh(\xi)\sin(\eta) \qquad \frac{\partial y}{\partial \eta} = (a)\sinh(\xi)\cos(\eta)$$

$$J\left(\frac{x,y}{\xi,\eta}\right) = \begin{vmatrix} (a)\sinh(\xi)\cos(\eta) & -(a)\cosh(\xi)\sin(\eta) \\ (a)\cosh(\xi)\sin(\eta) & (a)\sinh(\xi)\cos(\eta) \end{vmatrix}$$

$$= a^2\sinh^2(\xi)\cos^2(\eta) + a^2\cosh^2(\xi)\sin^2(\eta)$$

$$= a^2\left[\sinh^2(\xi)\left\{\frac{1+\cos(2\eta)}{2}\right\} + \cosh^2(\xi)\left\{\frac{1-\cos(2\eta)}{2}\right\}\right]$$

$$= \frac{a^2}{2}\left[\sinh^2(\xi) + \cosh^2(\xi) - \cos(2\eta)\left\{\cosh^2(\xi) - \sinh^2(\xi)\right\}\right]$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \cosh^2(x) - \sinh^2(x) = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \qquad \cosh^2(x) + \sinh^2(x) = \cosh(2x)$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$= \frac{a^2}{2}\left[\cosh(2\xi) - \cos(2\eta)\right]$$

Verify $JJ' = 1$ if $x = u(1-v),\ y = uv$.

Ans: Let $J = J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u - (-uv) = u$$

X. Find u and v in terms of x and y

$x = u - uv \implies x = u - y \implies u = x + y$

$y = uv \implies v = \frac{y}{u} = \frac{y}{x+y}$

$u = x + y \qquad v = \frac{y}{x+y}$

$$J' = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$u_x = 1 \qquad u_y = 1 \qquad v_x = y\left\{\frac{-1}{(x+y)^2}\right\} = \frac{-y}{(x+y)^2}$

$$v_y = \frac{(x+y) - y(0+1)}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$J' = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{x}{(x+y)^2} - \left\{\frac{-y}{(x+y)^2}\right\}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$JJ' = u\frac{1}{u} = 1$$

Verify $J J' = 1$ if $x = e^u \cos v,\ y = e^u \sin v$. where $J = J\left(\frac{x, y}{u, v}\right)$ & $J' = J\left(\frac{u, v}{x, y}\right)$

Ans:-

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} e^u \cos(v) & e^u\{-\sin(v)\} \\ e^u \sin(v) & e^u \cos(v) \end{vmatrix}$$

$$= (e^u)^2 \cos^2(v) - \{-(e^u)^2 \sin^2(v)\}$$

$$= e^{2u}\{\cos^2(v) + \sin^2(v)\} = e^{2u}$$

$u = u(x, y) = ?$ $\qquad v = v(x, y) = ?$

$$\frac{y}{x} = \tan(v) \implies v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x^2 + y^2 = (e^u)^2 \cos^2(v) + (e^u)^2 \sin^2(v) = e^{2u}$$

$$2u = \log(x^2 + y^2) \implies u = \frac{1}{2}\log(x^2 + y^2)$$

$$J' = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} =$$

$$u_x = \frac{1}{2}\left[\frac{1}{x^2 + y^2} \times (2x + 0)\right] = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{y}{x^2 + y^2}$$

$$v_x = \frac{1}{1 + (y/x)^2} \times y\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$v_y = \frac{1}{1 + (y/x)^2} \times \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$J' = \begin{vmatrix} \dfrac{x}{x^2+y^2} & \dfrac{y}{x^2+y^2} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{x^2}{(x^2+y^2)^2} - \left\{ \frac{-y^2}{(x^2+y^2)^2} \right\}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$$

$$JJ' = (e^{2u})\frac{1}{e^{2u}} = 1$$

If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, then verify $JJ' = 1$ where $J = J\left(\frac{y_1, y_2, y_3}{x_1, x_2, x_3}\right)$ and $J' = J\left(\frac{x_1, x_2, x_3}{y_1, y_2, y_3}\right)$.

Ans:- $$J = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \dfrac{\partial y_1}{\partial x_3} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \\ \dfrac{\partial y_3}{\partial x_1} & \dfrac{\partial y_3}{\partial x_2} & \dfrac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2} \qquad \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1} \qquad \frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2} \qquad \frac{\partial y_2}{\partial x_2} = \frac{-x_3 x_1}{x_2^2} \qquad \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3} \qquad \frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3} \qquad \frac{\partial y_3}{\partial x_3} = \frac{-x_1 x_2}{x_3^2}$$

$$J = \begin{vmatrix} -\dfrac{x_2 x_3}{x_1^2} & \dfrac{x_3}{x_1} & \dfrac{x_2}{x_1} \\ \dfrac{x_3}{x_2} & -\dfrac{x_1 x_3}{x_2^2} & \dfrac{x_1}{x_2} \\ \dfrac{x_2}{x_3} & \dfrac{x_1}{x_3} & -\dfrac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$J = \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2x_3 & x_1x_3 & x_1x_2 \\ x_2x_3 & -x_1x_3 & x_1x_2 \\ x_2x_3 & x_1x_3 & -x_1x_2 \end{vmatrix} \left[\begin{array}{l} \because \text{ from } R_1 \longrightarrow \frac{1}{x_1^2} \\ R_2 \longrightarrow \frac{1}{x_2^2} \quad \text{from } R_3 \longrightarrow \frac{1}{x_3^2} \end{array} \right.$$

$$= \frac{(x_2x_3)(x_1x_3)(x_1x_2)}{(x_1x_2x_3)^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \left[\begin{array}{l} \because \text{ from } C_1 \longrightarrow x_2x_2 \\ C_2 \longrightarrow x_1x_3 \quad C_3 \longrightarrow x_1x_2 \end{array} \right.$$

$$= \frac{x_2^2 x_3^2 x_1^2}{(x_1x_2x_3)^2} \left\{ -1(0) - 1(-1-1) + 1(1-(-1)) \right\}$$

$$= 1\{0+2+2\} = 4$$

$$y_1 = \frac{x_2x_3}{x_1} \qquad y_2 = \frac{x_3x_1}{x_2} \qquad y_3 = \frac{x_1x_2}{x_3}$$

$$x_1 = x_1(y_1, y_2, y_3) = ? \qquad x_2 = x_2(y_1, y_2, y_3) = ? \qquad x_3 = x_3(y_1, y_2, y_3) = ?$$

$$y_1y_2 = x_3^2 \qquad y_2y_3 = x_1^2 \qquad y_1y_3 = x_2^2$$

$$x_1 = \pm\sqrt{y_2y_3} \qquad x_2 = \pm\sqrt{y_1y_3} \qquad x_3 = \pm\sqrt{y_1y_2}$$

Consider $x_1 = \sqrt{y_2y_3} \qquad x_2 = \sqrt{y_1y_3} \qquad x_3 = \sqrt{y_1y_2}$

$$J' = J\left(\frac{x_1, x_2, x_3}{y_1, y_2, y_3}\right) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$

$$J' = \begin{vmatrix} 0 & \frac{\sqrt{y_3}}{2\sqrt{y_2}} & \frac{\sqrt{y_2}}{2\sqrt{y_3}} \\ \frac{\sqrt{y_3}}{2\sqrt{y_1}} & 0 & \frac{\sqrt{y_1}}{2\sqrt{y_3}} \\ \frac{\sqrt{y_2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{y_2}} & 0 \end{vmatrix} = \frac{1}{2\sqrt{y_1}}\frac{1}{2\sqrt{y_2}}\frac{1}{2\sqrt{y_3}} \begin{vmatrix} 0 & \sqrt{y_3} & \sqrt{y_2} \\ \sqrt{y_3} & 0 & \sqrt{y_1} \\ \sqrt{y_2} & \sqrt{y_1} & 0 \end{vmatrix}$$

$$J' = \frac{1}{8\sqrt{y_1y_2y_3}}\left\{0 - \sqrt{y_3}\left(-\sqrt{y_1y_2}\right) + \sqrt{y_2}\left(\sqrt{y_3y_1}\right)\right\}$$

$$= \frac{1}{8\sqrt{y_1y_2y_3}}\, 2\sqrt{y_1y_2y_3} = \frac{1}{4}$$

[We could have used chain rule or any other triplet for x_1, x_2, x_3

$$JJ' = 4\left(\frac{1}{4}\right) = 1$$

Qn) If $x = \frac{u^2 - v^2}{2}$, $y = uv$, $z = w$. find $J\left(\frac{u,v,w}{x,y,z}\right)$

Ans:- If $J = J\left(\frac{x,y,z}{u,v,w}\right) \neq 0$ then

$$J' = J\left(\frac{u,v,w}{x,y,z}\right) = \frac{1}{J} \qquad \left[\because JJ' = 1\right.$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$J = u^2 + v^2 \neq 0$ $\therefore$ Reqd Ans $J' = \frac{1}{u^2 + v^2}$

Property

If $u = u(x,y)$ $v = v(x,y)$ and

$x = x(r,s)$, $y = y(r,s)$

$$\boxed{J\left(\frac{u,v}{r,s}\right) = J\left(\frac{u,v}{x,y}\right) J\left(\frac{x,y}{r,s}\right)}$$

Techniques

$u \longrightarrow (x,y) \longrightarrow (r,s) \implies \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ $|||^{ly}$ for $\frac{\partial u}{\partial s}$

$v \longrightarrow (x,y) \longrightarrow (r,s)$

If $u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = r\cos\theta,\ y = r\sin\theta$. find $J\left(\frac{u,v}{r,\theta}\right)$

Ans: $(u,v) \longrightarrow (x,y) \longrightarrow (r,\theta)$

$$J\left(\frac{u,v}{r,\theta}\right) = J\left(\frac{u,v}{x,y}\right) \times J\left(\frac{x,y}{r,\theta}\right)$$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} = -4xy - (-16xy)$$

$$J\left(\frac{u,v}{x,y}\right) = 12xy$$

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= r\cos^2(\theta) - \{-r\sin^2\theta\} = r\{\cos^2(\theta) + \sin^2(\theta)\} = r$$

$$J\left(\frac{u,v}{r,\theta}\right) = 12xy(r) = 12r\cos(\theta)\, r\sin(\theta)\, r = 6r^3\sin(2\theta)$$

Find $\frac{\partial(X,Y)}{\partial(x,y)}$ where $X = u^2v$, $Y = uv^2$ and $u = x^2 - y^2$, $v = yx$

Ans: $(X,Y) \longrightarrow (u,v) \longrightarrow (x,y)$

$$\frac{\partial(X,Y)}{\partial(x,y)} = \frac{\partial(X,Y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)}$$

$$\frac{\partial(X,Y)}{\partial(u,v)} = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix} = \begin{vmatrix} 2uv & u^2 \\ v^2 & 2uv \end{vmatrix} = 4u^2v^2 - u^2v^2 = 3u^2v^2$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2 - (-2y^2) = 2(x^2+y^2)$$

$$\frac{\partial(X,Y)}{\partial(x,y)} = (3u^2v^2)(2)(x^2+y^2) = 6(x^2-y^2)^2(xy)^2(x^2+y^2)$$

HW

Qn) If $x=\sqrt{vw}$, $y=\sqrt{wu}$, $z=\sqrt{uv}$ and $u=r\sin\theta\cos\phi, v=r\sin\theta\sin\phi$, $w=r\cos\theta$, calculate $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

Qn) If $u=x\sqrt{1-y^2}+y\sqrt{1-x^2}$, $v=\sin^{-1}x+\sin^{-1}y$, show that u & v are functionally dependent and find the functional relationship.

$v=\alpha+\beta$ $\quad\because \sin^{-1}(x)=\alpha \Rightarrow \sin(\alpha)=x \quad |||^{ly}\ \sin\beta=y$

$$\sin(v)=\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

$$=x\left\{\sqrt{1-\sin^2\beta}\right\}+\sqrt{1-\sin^2\alpha}\;y$$

$$=x\sqrt{1-y^2}+\sqrt{1-x^2}\;y=u$$

$u=\sin(v)$

Part A $\longrightarrow$ Compulsory

1 $\longrightarrow$ 5 marks

Part B $\longrightarrow$ Compulsory

2
3
4

} 5 mark each

Part C

5 a or b $\longrightarrow$ 6 marks

6 a or b $\longrightarrow$ 7 marks

7 a or b — 7 marks

40 marks $\longrightarrow$ 75 min

Taylor's series for functions of 2 variables

Recollect : Taylor series of $f(x)$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$

$x - a = h$

$$f(x) = f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

Replace $h\frac{d}{dx}\Big|_{x=a} = \Delta \quad h^2\frac{d^2}{dx^2} = \Delta^2 \quad h^3\frac{d^3}{dx^3} = \Delta^3 \quad \cdots \quad h^n\frac{d^n}{dx^n} = \Delta^n$

$$f(x) = f(a) + \Delta f(a) + \frac{\Delta^2}{2!}f(a) + \frac{\Delta^3 f(a)}{3!} + \cdots \longrightarrow \text{Compact form}$$

Taylor series of $f(x,y)$

$x = a+h \quad y = b+k \qquad f(a+h, b+k) =$ powers of h & k

$$f(x,y) = f(a+h, b+k)$$
$$= f(a,b) + \Delta f(a,b) + \frac{\Delta^2 f(a,b)}{2!} + \frac{\Delta^3 f(a,b)}{3!} + \cdots$$

$$\left.\begin{aligned}
\Delta &= h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} = (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \\
\Delta^2 &= h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2} \\
\Delta^3 &= h^3\frac{\partial^3}{\partial x^3} + 3h^2k\frac{\partial^3}{\partial x^2\partial y} + 3hk^2\frac{\partial^3}{\partial x\partial y^2} + k^3\frac{\partial^3}{\partial y^3} \\
&\vdots
\end{aligned}\right\} \quad \begin{aligned} h &= x-a \\ k &= y-b \end{aligned}$$

If $(a,b) = (0,0)$ then Taylor's series reduces to Maclaurin's series

$$f(x,y) = f(0,0) + \Delta f(0,0) + \frac{\Delta^2 f(0,0)}{2!} + \frac{\Delta^3 f(0,0)}{3!} + \cdots$$

Expand $f(x,y)=x^2+xy+y^2$ in powers of $(x-1)$ and $(y-2)$.

Ans:- Taylor's series

$$f(x,y)=f(a,b)+\frac{\Delta f(a,b)}{1!}+\frac{\Delta^2 f(a,b)}{2!}+\frac{\Delta^3 f(a,b)}{3!}+\cdots$$

$$\Delta = h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y} \qquad h=(x-1) \quad k=y-2 \Rightarrow a=1 \quad b=2$$

$$f(x,y)=x^2+xy+y^2 \qquad f(1,2)=7$$

$$f_x=\frac{\partial f}{\partial x}=2x+y+0 \qquad f_x(1,2)=4$$

$$f_y=\frac{\partial f}{\partial y}=0+x+2y \qquad f_y(1,2)=5$$

$$\Delta f(1,2)=h\,f_x(1,2)+k\,f_y(1,2) = h(4)+k(5)$$

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(f_x) = 2$$

$$\Delta = A+B$$
$$\Delta^2 = A^2+2AB+B^2$$
$$=h^2\frac{\partial^2}{\partial x^2}+2hk\frac{\partial^2}{\partial x\partial y}+k^2\frac{\partial^2}{\partial y^2}$$

$$f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}(f_y) \text{ or } \frac{\partial}{\partial y}(f_x)=1$$

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(f_y)=2$$

$$\Delta^2 f(1,2)=h^2 f_{xx}(1,2)+2hk\,f_{xy}(1,2)+k^2 f_{yy}(1,2)$$
$$=h^2(2)+2hk(1)+k^2(2)$$

$$f(x,y)=7+4h+5k+\frac{1}{2!}\{2h^2+2hk+2k^2\}$$

At this stage replace $h=x-1$ $\quad k=y-2$

$$f(x,y)=7+4(x-1)+5(y-2)+(x-1)^2+(x-1)(y-2)+(y-2)^2$$

Express $f(x,y) = (1+x+y)^{-1}$ in powers of $(x-1)$ and $(y-1)$.

Ans:- Taylor series is $\quad [\because a=1 \quad b=1$

$$f(x,y) = f(1,1) + \Delta f(1,1) + \frac{\Delta^2 f(1,1)}{2!} + \frac{\Delta^3 f(1,1)}{3!} + \cdots$$

$$f(x,y) = \frac{1}{1+x+y} \qquad f(1,1) = \frac{1}{3}$$

$$\Delta = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \qquad \Delta^n = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

$$f_x = \frac{-1}{(1+x+y)^2}\{0+1+0\} \qquad f_x(1,1) = \frac{-1}{(1+1+1)^2} = \frac{-1}{9}$$

$$f_y = \frac{-1}{(1+x+y)^2}\{0+0+1\} \qquad f_y(1,1) = \frac{-1}{(1+1+1)^2} = \frac{-1}{9}$$

$$\Delta f(1,1) = h\left(-\frac{1}{9}\right) + k\left(-\frac{1}{9}\right) = -\frac{1}{9}(h+k)$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{-(-2)}{(1+x+y)^3}(0+1+0) = \frac{2}{(1+x+y)^3} \qquad f_{xx}(1,1) = \frac{2}{27}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{-(-2)}{(1+x+y)^3}\{0+0+1\} = \frac{2}{(1+x+y)^3} \qquad f_{xy}(1,1) = \frac{2}{27}$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{2}{(1+x+y)^3} \qquad f_{yy}(1,1) = \frac{2}{27}$$

$$\Delta^2 f(1,1) = h^2\left(\frac{2}{27}\right) + 2hk\frac{2}{27} + k^2\left(\frac{2}{27}\right) = \frac{2}{27}\{h^2+2hk+k^2\}$$

$$f(x,y) = \frac{1}{3} - \frac{1}{9}(h+k) + \frac{1}{2!}\left\{\frac{2}{27}(h^2+2hk+k^2)\right\}$$

$$= \frac{1}{3} - \frac{1}{9}\{(x-1)+(y-1)\} + \frac{1}{27}\{(x-1)^2 + 2(x-1)(y-1) + (y-1)^2\} + \cdots$$

No further simplication should be done

Approximate the value of $(1.1)^{1.1}$ using Taylor's series

Ans:- let $f(x,y) = x^y$ Req'd Ans : $(1.1)^{1.1} = 1.11053$ approx

Express $f(x,y)$ in powers of $\underbrace{(x-1)}_{a}$ & $\underbrace{(y-1)}_{b}$

$$f(x,y) = f(1,1) + \frac{\Delta f(1,1)}{1!} + \frac{\Delta^2 f(1,1)}{2!} + \frac{\Delta^3 f(1,1)}{3!} + \cdots\cdots$$

$$\Delta = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \qquad \Delta^n = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

$f(x,y) = x^y$ $\qquad f(1,1) = 1^1 = 1$

$f_x = y x^{y-1}$ [Similar to x^n $\qquad f_x(1,1) = 1(1^0) = 1$

$f_y = x^y \log(x)$ [Similar to a^y

$f_y = f \log(x)$ $\qquad f_y(1,1) = f(1,1)\log(1) = 0$

$$\Delta f(1,1) = h f_x(1,1) + k f_y(1,1) = h(1) + k(0) = h$$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = y(y-1)x^{y-2}$ $\qquad f_{xx}(1,1) = 1(0)1^{-1} = 0$

$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\{f \log x\} = f_x \log(x) + f\left(\frac{1}{x}\right)$ $\qquad f_{xy}(1,1) = 1(0) + 1\left(\frac{1}{1}\right) = 1$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = f_y \log(x)$ $\qquad f_{yy}(1,1) = 0(0) = 0$

$$\Delta^2 f(1,1) = h^2 f_{xx}(1,1) + 2hk f_{xy}(1,1) + k^2 f_{yy}(1,1)$$
$$= h^2(0) + 2hk(1) + k^2(0) = 2hk$$

$f_{xxx} = \frac{\partial}{\partial x}\{f_{xx}\} = y(y-1)(y-2)x^{y-3}$ $\qquad f_{xxx}(1,1) = 1(0)(-1)1^{-2} = 0$

$f_{xxy} = \frac{\partial}{\partial y}(f_{xx}) = \frac{\partial}{\partial y}\{(y^2-y)x^{y-2}\} = (2y-1)x^{y-2} + (y^2-y)x^{y-2}\log(x)$

$f_{xxy}(1,1) = 1(1)^{-1} + 0(1^{-1})(0) = 1$

$f_{xyy} = \frac{\partial}{\partial y}(f_{xy}) = \frac{\partial}{\partial x}(f_{yy}) = \frac{\partial}{\partial x}(f_y \log x) = f_{yx}\log(x) + f_y \frac{1}{x}$

$f_{xyy}(1,1) = 1(0) + (0)(1) = 0$

$f_{yyy} = \frac{\partial}{\partial y}(f_{yy}) = f_{yy}\log(x)$ $\qquad f_{yyy}(1,1) = 0(0) = 0$

$\Delta^3 f(1,1) = h^3 f_{xxx}(1,1) + 3h^2k f_{xxy}(1,1) + 3hk^2 f_{xyy}(1,1) + k^3 f_{yyy}(1,1)$

$= 0 + 3h^2k + 0 + 0 = 3h^2k$

$(a+b)^0 \longrightarrow 1$
$(a+b)^1 \longrightarrow 1 \quad 1$
$(a+b)^2 \longrightarrow 1 \quad 2 \quad 1$
$(a+b)^3 \longrightarrow 1 \quad 3 \quad 3 \quad 1$
$1 \quad 4 \quad 6 \quad 4 \quad 1$

$f(x,y) = x^y = 1 + h + \frac{1}{2!}\{2hk\} + \frac{1}{3!}3h^2k$

$x^y = 1 + (x-1) + (x-1)(y-1) + \frac{(x-1)^2(y-1)}{2} + \cdots$

$x = 1.1 \quad y = 1.1$

$(1.1)^{1.1} = 1 + 0.1 + (0.1)(0.1) + \frac{(0.1)^2(0.1)}{2} + \cdots$
$= 1.1105$

Qn) Expand $f(x,y) = xy^2 + \cos xy$ in powers of $(x-1)$ and $(y - \pi/2)$. $\Longrightarrow a = 1 \quad b = \frac{\pi}{2}$

Ans: Taylor's series: $f(x,y) = f(1,\frac{\pi}{2}) + \frac{\Delta f(1,\frac{\pi}{2})}{1!} + \frac{\Delta^2 f(1,\frac{\pi}{2})}{2!} + \frac{\Delta^3 f(1,\frac{\pi}{2})}{3!} + \cdots$

$f(x,y) = xy^2 + \cos(xy)$ $\qquad f(1,\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^2 + \cos\frac{\pi}{2} = \frac{\pi^2}{4}$

$f_x = \frac{\partial f}{\partial x} = y^2 - \sin(xy).y$ $\qquad f_x(1,\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^2 - 1.\frac{\pi}{2} = \frac{\pi^2}{4} - \frac{\pi}{2}$

$f_y = \frac{\partial f}{\partial y} = x2y - \sin(xy).x$ $\qquad f_y(1,\frac{\pi}{2}) = \pi - (1)1 = \pi - 1$

$\Delta f(1,\frac{\pi}{2}) = h f_x(1,\frac{\pi}{2}) + k f_y(1,\frac{\pi}{2}) = \left(\frac{\pi^2}{4} - \frac{\pi}{2}\right)h + (\pi-1)k$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = 0 - \cos(xy)\,y^2$ $\qquad f_{xx}(1,\frac{\pi}{2}) = -(0)\left(\frac{\pi^2}{4}\right) = 0$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\{y^2 - y\sin(xy)\} = 2y - \{\sin(xy) + y\cos(xy).x\}$$

$$f_{xy} = 2y - \sin(xy) - xy\cos(xy) \qquad f_{xy}(1, \tfrac{\pi}{2}) = \pi - 1 - \tfrac{\pi}{2}(0) = \pi - 1$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = 2x - \cos(xy).x^2 \qquad f_{yy}(1, \tfrac{\pi}{2}) = 2 - 0(1) = 2$$

$$\Delta^2 f(1,1) = h^2 f_{xx}(1, \tfrac{\pi}{2}) + 2hk f_{xy}(1, \pi/2) + k^2 f_{yy}(1, \tfrac{\pi}{2})$$

$$= 0 + 2hk(\pi - 1) + k^2(2)$$

$$f(x,y) = \frac{\pi^2}{4} + \left(\frac{\pi^2}{4} - \frac{\pi}{2}\right)h + (\pi - 1)k + \frac{1}{2!}\left\{2hk(\pi-1) + 2k^2\right\} + \cdots\cdots$$

$$h = (x-1) \qquad k = y - \frac{\pi}{2}$$

$$f(x,y) = \frac{\pi^2}{4} + \left(\frac{\pi^2}{4} - \frac{\pi}{2}\right)(x-1) + (\pi-1)\left(y - \frac{\pi}{2}\right) + (x-1)\left(y - \frac{\pi}{2}\right)(\pi - 1) + \left(y - \frac{\pi}{2}\right)^2 + \cdots$$

Qn) Expand $e^y \log_e(1+x)$ in powers of x and y up to third degree terms.

Ans:- Maclaurin's series = Taylor series about $(0,0)$

$$f(x,y) = f(0,0) + \frac{\Delta f(0,0)}{1!} + \frac{\Delta^2 f(0,0)}{2!} + \frac{\Delta^3 f(0,0)}{3!} + \cdots\cdots$$

$$f(x,y) = e^y \log(1+x) \qquad f(0,0) = e^0 \log(1) = 0$$

$$f_x = \frac{\partial}{\partial x}(f) = e^y \frac{1}{1+x} \qquad f_x(0,0) = e^0 \frac{1}{1+0} = 1$$

$$f_y = \frac{\partial f}{\partial y} = e^y \log(1+x) = f \qquad f_y(0,0) = f(0,0) = 0$$

$$\Delta f(0,0) = h f_x(0,0) + k f_y(0,0) = h + 0 = h$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = e^y\left\{\frac{-1}{(1+x)^2}\right\} \qquad f_{xx}(0,0) = e^0\left\{\frac{-1}{(1+0)^2}\right\} = -1$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial x}(f_y) = \frac{\partial f}{\partial x} = f_x \qquad f_{xy}(0,0) = f_x(0,0) = 1$$

$$f_{yy} = \frac{\partial(f_y)}{\partial y} = \frac{\partial f}{\partial y} = f_y \qquad f_{yy}(0,0) = f_y(0,0) = 0$$

$$\Delta^2 f(0,0) = h^2 f_{xx}(0,0) + 2hk f_{xy}(0,0) + k^2 f_{yy}(0,0)$$

$$= h^2(-1) + 2hk(1) + 0 = 2hk - h^2$$

$$f_{xxx} = \frac{\partial}{\partial x}(f_{xx}) = -e^y \frac{(-2)}{(1+x)^3} = \frac{2e^y}{(1+x)^3} \qquad f_{xxx}(0,0) = \frac{2e^0}{(1+0)^3} = 2$$

$$f_{xxy} = \frac{\partial}{\partial y}(f_{xx}) = \frac{-1}{(1+x)^2} e^y = f_{xx} \qquad f_{xxy}(0,0) = f_{xx}(0,0) = -1$$

$$f_{xyy} = \frac{\partial}{\partial y}(f_{xy}) = \frac{\partial}{\partial y}(f_x) = f_{xy} \qquad f_{xyy}(0,0) = f_{xy}(0,0) = 1$$

$$f_{yyy} = \frac{\partial}{\partial y}(f_{yy}) = \frac{\partial}{\partial y}(f_y) = f_{yy} \qquad f_{yyy}(0,0) = f_{yy}(0,0) = 0$$

$$\Delta^3 f(0,0) = h^3 f_{xxx}(0,0) + 3h^2 k f_{xxy}(0,0) + 3hk^2 f_{xyy}(0,0) + k^3 f_{yyy}(0,0)$$

$$= h^3(2) + 3h^2 k(-1) + 3hk^2(1) + 0$$

$$f(x,y) = 0 + \frac{h}{1!} + \frac{2hk - h^2}{2!} + \frac{2h^3 - 3h^2k + 3hk^2}{3!}$$

$$h = (x - 0) = x \qquad k = y - 0 = y$$

$$f(x,y) = x + \frac{2xy - x^2}{2} + \frac{2x^3 - 3x^2y + 3xy^2}{6} + \cdots\cdots$$

Qn) Obtain the Maclaurin's series of $e^{ax}\sin(by)$

Ans:- $$f(x,y) = f(0,0) + \frac{\Delta f(0,0)}{1!} + \frac{\Delta^2 f(0,0)}{2!} + \frac{\Delta^3 f(0,0)}{3!} + \cdots$$

$$f(x,y) = e^{ax}\sin(by) \qquad f(0,0) = 0$$

$$f_x = \frac{\partial f}{\partial x} = ae^{ax}\sin(by) = af \qquad f_x(0,0) = 0$$

$$f_y = \frac{\partial f}{\partial y} = e^{ax}\cos(by)\,b \qquad f_y(0,0) = b$$

$$\Delta f(0,0) = hf_x(0,0) + kf_y(0,0) = kb$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(af) = af_x = a^2 f \qquad f_{xx}(0,0) = 0$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(af) = af_y \qquad f_{xy}(0,0) = a(b) = ab$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = e^{ax}\{-\sin(by)\}b^2$$

$$f_{yy} = -b^2 f \qquad f_{yy}(0,0) = 0$$

$$\Delta^2 f(0,0) = h^2 f_{xx}(0,0) + 2hk f_{xy}(0,0) + k^2 f_{yy}(0,0)$$
$$= 2hk(ab)$$

$$f_{xxx} = \frac{\partial}{\partial x}(f_{xx}) = \frac{\partial}{\partial x}(a^2 f) = a^2 f_x = a^3 f \qquad f_{xxx}(0,0) = 0$$

$$f_{xxy} = \frac{\partial}{\partial y}(f_{xx}) = \frac{\partial}{\partial y}(a^2 f) = a^2 f_y \qquad f_{xxy}(0,0) = a^2(b)$$

$$f_{xyy} = \frac{\partial}{\partial y}(f_{xy}) = \frac{\partial}{\partial x}(f_{yy}) = \frac{\partial}{\partial x}(-b^2 f) = -b^2 f_x \qquad f_{xyy}(0,0) = 0$$

$$f_{yyy} = \frac{\partial}{\partial y}(f_{yy}) = \frac{\partial}{\partial y}(-b^2 f) = -b^2 f_y \qquad f_{yyy}(0,0) = -b^2(b)$$

$$f_{yyy}(0,0) = -b^3$$

$$\Delta^3 f(0,0) = h^3 f_{xxx}(0,0) + 3h^2k f_{xxy}(0,0) + 3hk^2 f_{xyy}(0,0) + k^3 f_{yyy}(0,0)$$
$$= 3h^2k(a^2b) + k^3(-b^3)$$

$$f(x,y) = 0 + \frac{kb}{1!} + \frac{2hk\,ab}{2!} + \frac{3h^2ka^2b - k^3b^3}{3!}$$

$$h = x - a = x \qquad k = y - b = y \qquad (\because a = 0 = b \text{ for Maclaurin's series})$$

$$f(x,y) = by + abxy + \frac{3a^2bx^2y - b^3y^3}{6} + \cdots\cdots$$

Expand $f(x,y)=\cot^{-1}(xy)$ in powers of $(x+0.5)$ and $(y-2)$ and hence compute $f(-0.4,2.2)$

Ans:- $f(x,y)=-\frac{\pi}{4}-(x+0.5)+\frac{1}{4}(y-2)-(x+0.5)^2-\frac{1}{16}(y-2)^2+\cdots$

Put $x=-0.4$ and $y=2.2$ in RHS of Maclaurin series

$f(-0.4,2.2)=-\frac{\pi}{4}-(0.1)+\frac{0.2}{4}-(0.1)^2-\frac{(0.2)^2}{16}+\cdots$

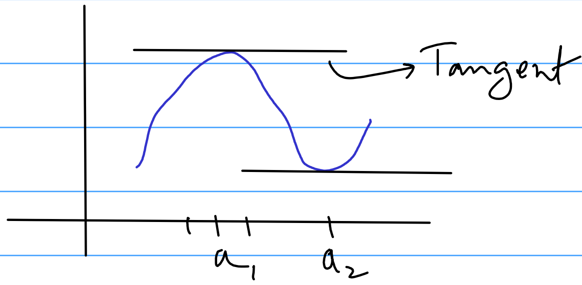
$=-0.8478982$

shift $\times 10^x$ to access value of π

Maxima and Minima for $f(x,y)$

Recollect: $y = f(x)$

Necessary condition for Maxima or minima : $f'(x) = 0$

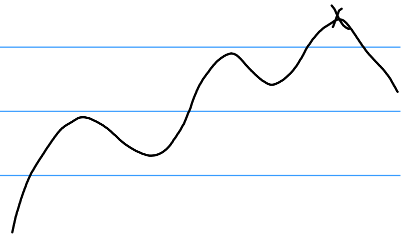


At maxima or minima slope of tangent $= 0$

$\therefore f'(x) = 0$

Maxima: $f(x) - f(a) < 0 \qquad x \in (a-h, a+h)$

Minima: $f(x) - f(a) > 0 \qquad x \in (a-h, a+h)$



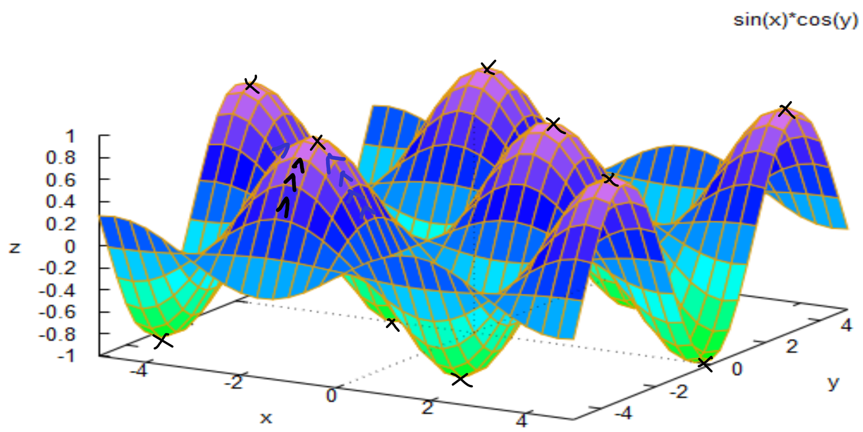
$$f(x) = f(a) + \frac{h f'(a)}{1!} + \frac{h^2 f''(a)}{2!} + O(h^3)$$

'a' is a point of extrema

$$f(x) - f(a) = \frac{h^2}{2!} f''(a) + \text{negligible}$$

$f(x) - f(a) < 0$ if $f''(a) < 0$

$f(x) - f(a) > 0$ if $f''(a) > 0$



wxmaxima

free and

Open source

Maxima

$$f(x,y) - f(a,b) < 0 \quad \forall \quad x \in (a-h, a+h) \quad y \in (b-k, b+k)$$

Minima

$$f(x,y) - f(a,b) > 0 \quad \forall \; x \in (a-h, a+h) \quad y \in (b-k, b+k)$$

At points of extrema tangent planes are parallel to XY plane (i.e $z=0$)

$$\Rightarrow \quad \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0 \qquad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0 \implies \Delta f(a,b) = 0$$

$$f(x,y) = f(a,b) + \frac{\Delta f(a,b)}{1!} + \frac{\Delta^2 f(a,b)}{2!} + \cdots$$

$$f(x,y) - f(a,b) = \frac{\Delta^2 f(a,b)}{2!} + \text{negligible}$$

$f(x,y) - f(a,b) < 0$ or > 0 depending on $\Delta^2 f(a,b)$

$$f(x,y) - f(a,b) = \frac{1}{2!}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]$$

$$= \frac{1}{2!}\left[h^2 r + 2hks + k^2 t\right]$$

$$= \frac{1}{2!\,r}\left[h^2 r^2 + 2hksr + k^2 rt\right]$$

$$= \frac{1}{2!\,r}\left[h^2 r^2 + 2(hr)(ks) + (ks)^2 - (ks)^2 + k^2 rt\right]$$

$$f(x,y) - f(a,b) = \frac{1}{2!\,\underset{*}{r}}\left[(hr+ks)^2 + k^2\underset{*}{\underline{(rt - s^2)}}\right]$$

Maxima

$f(x,y) - f(a,b) < 0$ if $r < 0$ and $(rt - s^2) > 0$

Mininima

$f(x,y) - f(a,b) > 0$ if $r > 0$ and $(rt - s^2) > 0$

$rt - s^2 < 0 \implies$ Saddle point

$rt - s^2 = 0 \implies$ Inconclusive

Extrema of $f(x,y)$

$\downarrow$

$f_x = ?,\ f_y = ?$

$\downarrow$

Solve $f_x = 0 = f_y$ to get (say) $(a,b) \rightarrow$ stationary points

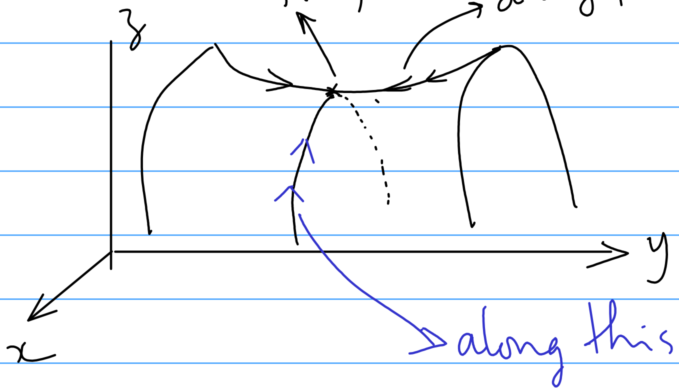
$\downarrow$

$rt - s^2$

$\downarrow$

- $(rt - s^2) > 0$
  - $\boxed{r < 0 \implies \text{Maxima}}$
  - $\boxed{r > 0 \implies \text{Minima}}$
- $rt - s^2 = 0 \implies$ Inconclusive
- $rt - s^2 < 0 \implies$ Saddle point

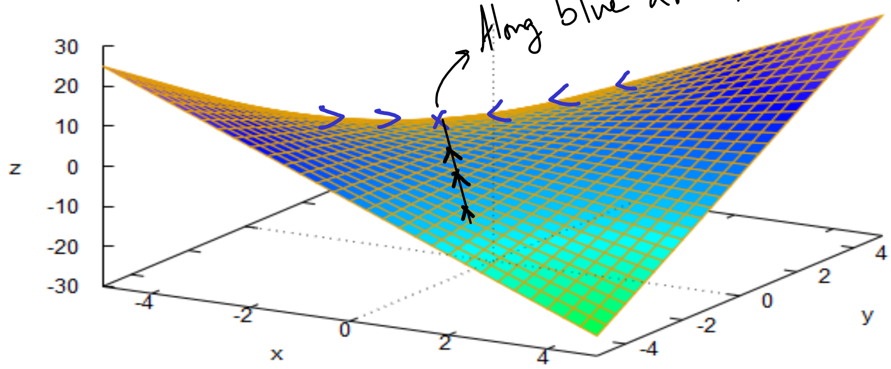
$f(a,b)$ along this curve (a,b) is point of minima



along this curve (a,b) is point of maxima

$z =$ x*y

Along blue arrow, minima and maxima along black arrow & hence it is called Saddle point



Discuss the nature of the stationary points on the surface $f(x,y) = x^3+y^3-3y-12x+20$

Ans:- $f(x,y) = x^3+y^3-3y-12x+20$

step 1:- Find f_x, f_y

$$f_x = \frac{\partial f}{\partial x} = 3x^2+0-0-12 = 3x^2-12$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2-3$$

step 2: find roots of $f_x = 0 = f_y$

$$3x^2-12=0 \Rightarrow x = \pm 2$$

$$3y^2-3=0 \Rightarrow y = \pm 1$$

$(2,1), (2,-1), (-2,1), (-2,-1) \rightarrow$ stationary points

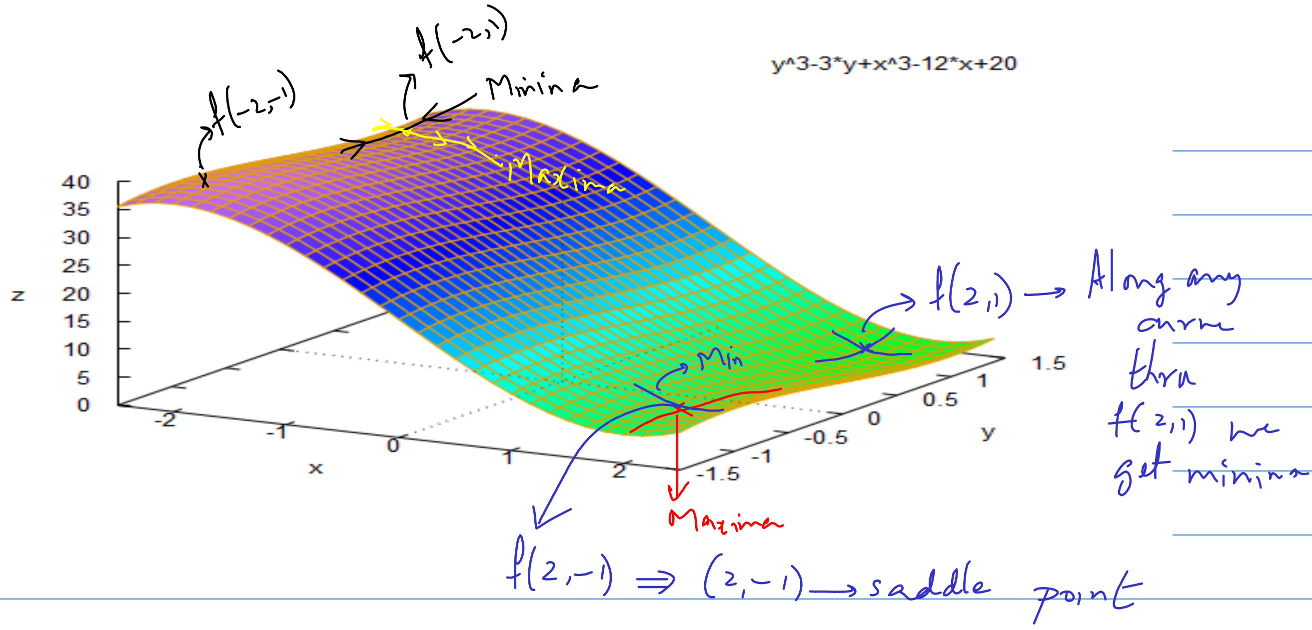
step 3:- 2nd order derivatives

$$r = f_{xx} = \frac{\partial f_x}{\partial x} = 6x$$

$$s = f_{xy} = \frac{\partial}{\partial y}(f_x) = 0$$

$$t = f_{yy} = \frac{\partial}{\partial y}(f_y) = 6y$$

stationary points (a,b)	r	s	t	$rt-s^2$	Nature of (a,b)	Inference
$(2,1)$	12	0	6	$72>0$	Extrema	Minima [$\because r>0$
$(2-1)$	12	0	-6	$-72<0$	Saddle	
$(-2,1)$	-12	0	6	$-72<0$	Saddle	
$(-2,-1)$	-12	0	-6	$72>0$	Extrema	Maxima [$\because r<0$



Qn) Find the extreme values of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Ans: $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Step 1: $f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x + 72 \longrightarrow (1)$

$f_y = \frac{\partial f}{\partial y} = 6xy - 30y \longrightarrow (2)$

Step 2: $f_x = 0 = f_y$

$3x^2 + 3y^2 - 30x + 72 = 0$

$\frac{f_x}{3} = x^2 + y^2 - 10x + 24 = 0 \longrightarrow (3)$

$6xy - 30y = 0 \Rightarrow \frac{f_y}{6} = y(x-5) = 0 \longrightarrow (4)$

(4) $\Rightarrow y = 0$ or $x = 5$

$y = 0$ in (3) $\Rightarrow x^2 - 10x + 24 = 0 \Rightarrow x = 4, 6$

$\Rightarrow (4,0)$ and $(6,0)$ are the stationary points

$x = 5$ in (3) $\Rightarrow 25 + y^2 - 50 + 24 = 0 \Rightarrow y = 1, -1$

$\Rightarrow (5,1)$ and $(5,-1)$ are also stationary points step 3: $r, s, t = ?$

$$r = f_{xx} = \frac{\partial}{\partial x}(f_x) = 6x - 30$$

$$s = f_{xy} = \frac{\partial}{\partial y}(f_x) = 6y$$

[Using eqn (1) to find f_{xx} and f_{xy}

$$t = f_{yy} = 6x - 30$$

Use eqn (2) to find t

Stationary pts (a,b)	r	s	t	$rt - s^2$	Nature of (a,b)	Inference
(4,0)	-6	0	-6	36 > 0	extrema	Maxima [as $r<0$
(6,0)	6	0	6	36 > 0	extrema	Minima [as $r>0$
(5,1)	0	6	0	-36 < 0	Saddle point	
(5,-1)	0	-6	0	-36 < 0	Saddle point	

$$f_{max} = f(4,0) = 112$$

$$f_{min} = f(6,0) = 108$$

Discuss the nature of the stationary for the Surface $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Ans:- $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

step 1 [

$$f_x = \frac{\partial f}{\partial x} = 4x^3 + 0 - 4x + 4y \longrightarrow (1)$$

$$f_y = \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y \longrightarrow (2)$$

step 2 :- $f_x = 0 = f_y$

$$4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0 \longrightarrow (3)$$

$$4y^3 + 4x - 4y = 0 \Longrightarrow y^3 + x - y = 0 \longrightarrow (4)$$

$$③ + ④ \Rightarrow x^3 + y^3 = 0$$
$$(x+y)(x^2 - xy + y^2) = 0$$
$$\Rightarrow x = -y \quad \text{or} \quad y = -x$$
$$y = -x \text{ in } ③ \Rightarrow x^3 - x[-x] = 0 \Rightarrow x(x^2-2) = 0$$
$$x = 0, \sqrt{2}, -\sqrt{2}$$
$$y = -x \Rightarrow \begin{cases} \text{when } x = 0, y = 0 \Rightarrow (0,0) \\ \text{when } x = \sqrt{2}, y = -\sqrt{2} \Rightarrow (\sqrt{2}, -\sqrt{2}) \\ \text{when } x = -\sqrt{2}, y = \sqrt{2} \Rightarrow (-\sqrt{2}, \sqrt{2}) \end{cases}$$

$r = f_{xx} = \frac{\partial}{\partial x}(f_x) = 12x^2 - 4$ [using eqn ①

$$s = f_{xy} = \frac{\partial}{\partial y}(f_x) = 4$$
$$t = f_{yy} = \frac{\partial}{\partial y}(f_y) = 12y^2 - 4$$

Stationary pts (a,b)	r	s	t	$rt - s^2$	Nature of (a,b)	Inference
$(0,0)$	-4	4	-4	0	Inconclusive	Further tests needed
$(\sqrt{2}, -\sqrt{2})$	20	4	20	$384 > 0$	Extrema	Minima [as $r > 0$
$(-\sqrt{2}, \sqrt{2})$	20	4	20	$384 > 0$	Extrema	Minima (as $r > 0$

$$f_{min} = f(\sqrt{2}, -\sqrt{2}) = f(-\sqrt{2}, \sqrt{2}) = -8$$

Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$

Ans:- Step 1: $f_x = \frac{\partial f}{\partial x} = y^2(3x^2 - 4x^3 - 3x^2y) \longrightarrow (1)$

$f_y = \frac{\partial f}{\partial y} = x^3(2y - x2y - 3y^2) \longrightarrow (2)$

Step 2:- Stationary points? $f_x = 0 = f_y$

$(1) \Rightarrow \quad y^2x^2(3 - 4x - 3y) = 0 \longrightarrow (3)$

$(2) \Rightarrow \quad x^3y(2 - 2x - 3y) = 0 \longrightarrow (4)$

$(3) \Rightarrow \quad y = 0, \quad x = 0, \quad 3 - 4x - 3y = 0$

$(4) \Rightarrow \quad x = 0, \quad y = 0, \quad 2 - 2x - 3y = 0$

7 points usual process

$y = 0$ statisfies both (3) and (4) $\Rightarrow$ Any point on x-axis is a stationary point i.e. $(a, 0)$

$x = 0$ statisfies both (3) and (4) $\Rightarrow$ Any point on y-axis is a stationary point i.e. $(0, b)$

$3 - 4x - 3y = 0 \Rightarrow 4x + 3y = 3$

$2 - 2x - 3y = 0 \Rightarrow 2x + 3y = 2$

$2x = 1 \Rightarrow x = \frac{1}{2}$

$x = \frac{1}{2} \quad y = \frac{1}{3} \qquad \left(\frac{1}{2}, \frac{1}{3}\right)$

$r = f_{xx} = \frac{\partial}{\partial x}(f_x) = y^2(6x - 12x^2 - 6xy)$

$= y^2x(6 - 12x - 6y)$

$s = f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial x}(f_y) = 3x^2 2y - 4x^3 2y - 9x^2y^2$

$t = f_{yy} = \frac{\partial}{\partial y}(f_y) = x^3(2 - 2x - 6y)$

Stationary pts (a, b)	r	s	t	$rt - s^2$	Nature of (a,b)	Inference
$(a, 0)$	0	0	$a^3(2-2a)$	0	Inconclusive Test	Further test
$(0, b)$	0	0	0	0	Inconclusive Test	Further test
$\left(\frac{1}{2}, \frac{1}{3}\right)$	$-\frac{1}{9}$	$-\frac{1}{12}$	$-\frac{1}{8}$	$\frac{1}{144} > 0$	Extrema	Pt of Maxima

$f_{max} = f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$

Divide 120 into three parts so that the sum of their products taken two at a time is maximum.

Ans:-

$\underbrace{x \mid y \mid z}_{120}$

$x + y + z = 120 \rightarrow$ Condition ①

$xy + yz + zx = F$

Task: Find x, y, z such that F is maximum

$z = 120 - x - y$ [from ①

$F(x,y) = xy + y(120 - x - y) + x(120 - x - y)$

$F(x,y) = 120y + 120x - xy - y^2 - x^2 \rightarrow$ ②

Step 1: $F_x = \frac{\partial F}{\partial x} = 120 - y - 2x \rightarrow$ ③

$F_y = \frac{\partial F}{\partial y} = 120 - x - 2y \rightarrow$ ④

step 2 :- $F_x = F_y = 0$

$$120 - y - 2x = 0 \Rightarrow 2x + y = 120$$

$$120 - x - 2y = 0 \Rightarrow \underline{x + 2y = 120}$$

$$4x + 2y = 240$$
$$x + 2y = 120$$
$$\overline{3x = 120} \Rightarrow x = 40 \Rightarrow y = 40$$

$(40, 40)$

step 3 : $r = f_{xx} = \frac{\partial}{\partial x}(f_x) = -2$

$s = f_{xy} = \frac{\partial}{\partial y}(f_x) = -1$

$t = f_{yy} = \frac{\partial}{\partial y}(f_y) = -2$

At $(40,40)$ $\quad rt - s^2 = 4 - (-1)^2 = 3 > 0$

$r < 0 \Rightarrow$ Pt of Maxima

$x + y + z = 120 \Rightarrow z = 40$

$xy + yz + zx$ is Max when $x = 40 = y = z$
given $(x + y + z) = 120$

Find the shortest distance from origin to the surface $xyz^2 = 2$

Ans:- let $P(x, y, z)$ be any point on the surface $xyz^2 = 2$

distance of P from $(0,0,0)$ is, $d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$$d = \sqrt{x^2+y^2+z^2}$$

Task find d_{min} with $xyz^2 = 2$

$$F = d^2 \qquad d_{min} = \sqrt{F_{min}}.$$

$$F = x^2+y^2+z^2 \qquad \text{with} \quad xyz^2=2 \Rightarrow z^2 = \frac{2}{xy}$$

$$F(x,y) = x^2+y^2+\frac{2}{xy}$$

Step 1: $\frac{\partial F}{\partial x} = F_x = 2x + \frac{2}{y}\left(-\frac{1}{x^2}\right) = 2x - \frac{2}{x^2y} \rightarrow (1)$

$$\frac{\partial F}{\partial y} = F_y = 2y - \frac{2}{xy^2} \longrightarrow (2)$$

step 2: $F_x = 0 = F_y$

$$\frac{2(x^3y-1)}{x^2y} = 0 \Rightarrow x^3y-1=0 \rightarrow (3)$$

$$\frac{2(xy^3-1)}{xy^2} = 0 = xy^3-1=0 \longrightarrow (4)$$

$$(3)-(4) \Longrightarrow x^3y - xy^3 = 0$$

$$\Longrightarrow xy(x^2-y^2)=0$$

$$x=0, \quad y=0 \quad x^2-y^2=0$$

$x=0,\ y=0,\ y=\pm x \qquad \left[y^2 = x^2 \Rightarrow y = \pm\sqrt{x^2} \right.$
$y = \pm\sqrt{x}$

$x=0$ or $y=0$ does not satisfy ③ or ④

$y=x$ in ③ (or 4) $\quad x^4-1=0$

$\Rightarrow x = 1, -1 \quad \left[i, -i \right.$ are imaginary

$\Rightarrow (1,1), (-1,-1)$

$y=-x$ in ③ $\Rightarrow -x^4-1=0$

$x^4 = -1 \Longrightarrow$ imaginary roots

$\therefore y=-x$ is rejected

$$r = F_{xx} = \frac{\partial}{\partial x}(F_x) = 2 - \frac{2}{y}\left(\frac{-2}{x^3}\right) = 2 + \frac{4}{x^3 y}$$

$$s = F_{xy} = \frac{\partial}{\partial y}(F_x) = 0 - \frac{2}{x^2}\left(-\frac{1}{y^2}\right) = \frac{2}{x^2y^2}$$

$$t = F_{yy} = \frac{\partial}{\partial y}(F_y) = 2 - \frac{2}{x}\left(\frac{-2}{y^3}\right) = 2 + \frac{4}{xy^3}$$

Stationary pt (a,b)	r	s	t	$rt-s^2$	Nature of (a,b)	Inference
$(1,1)$	6	2	6	$32>0$	Extrema	Minima
$(-1,-1)$	6	2	6	$32>0$	Extrema	

$xyz^2 = 2 \Longrightarrow z^2 = \frac{2}{xy} \Rightarrow z = \pm\sqrt{\frac{2}{xy}}$

at $x=1, y=1 \quad z = \pm\sqrt{2} \Rightarrow (1,1,\sqrt{2}), (1,1,-\sqrt{2})$

at $x=-1, y=-1 \quad z = \pm\sqrt{2} \Rightarrow (-1,-1,\sqrt{2}), (-1,-1,-\sqrt{2})$

A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y)=x^2+2y^2-x$. Find the coldest point on the plate.

Ans:- $u(x,y) = x^2 + 2y^2 - x$

We need (x,y) where $u(x,y)$ is minimum

step1 $\begin{cases} \dfrac{\partial u}{\partial x} = 2x - 1 = u_x \\ \dfrac{\partial u}{\partial y} = 4y = u_y \end{cases}$

step2: $\dfrac{\partial u}{\partial x} = 0 = \dfrac{\partial u}{\partial y}$

$2x - 1 = 0$ $\qquad$ $4y = 0$

$x = 1/2$ $\qquad$ $y = 0$ $\qquad$ $\left(\frac{1}{2}, 0\right)$

$r = \dfrac{\partial^2 u}{\partial x^2} = u_{xx} = \dfrac{\partial}{\partial x}(u_x) = 2$

$s = \dfrac{\partial^2 u}{\partial x \partial y} = u_{xy} = \dfrac{\partial}{\partial y}(u_x) = 0$

$t = \dfrac{\partial^2 u}{\partial y^2} = u_{yy} = \dfrac{\partial}{\partial y}(u_y) = 4$

$rt - s^2 = 8 > 0 \implies \left(\frac{1}{2}, 0\right)$ is a point of extrema

$r = 2 > 0 \implies \left(\frac{1}{2}, 0\right)$ is a point of minima

$\implies \left(\frac{1}{2}, 0\right)$ is the coldest point

$u_{min} = u\left(\frac{1}{2}, 0\right) = \frac{1}{4} + 0 - \frac{1}{2} = -\frac{1}{4}$ units

Qn) The temperature T at any point (x, y, z) in space is $T(x, y, z) = kxyz^2$ where k is a constant (>0). Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Ans: $T(x,y,z) = kxyz^2 \longrightarrow (1)$

condition: $x^2+y^2+z^2 = a^2$

Task: T_{max} on the surface?

$z^2 = a^2 - x^2 - y^2 \longrightarrow (2)$ [$\because$ using condition

$f(x,y) = T^*(x,y) = kxy\{a^2 - x^2 - y^2\}$

Task: $f_{max} = ?$

Step 1: $f_x = ?$ $f_y = ?$

$f_x = \frac{\partial f}{\partial x} = ky\{a^2 - 3x^2 - y^2\} \rightarrow (3)$ [$\because f = ky\{a^2x - x^3 - y^2x\}$

$f_y = \frac{\partial f}{\partial y} = kx\{a^2 - x^2 - 3y^2\} \rightarrow (4)$ [$\because f = kx\{a^2y - x^2y - y^3\}$

Step 2: Stationary points

$f_x = 0 \implies ky\{a^2 - 3x^2 - y^2\} = 0 \rightarrow (5)$

$(5) \implies y = 0$ or $(a^2 - 3x^2 - y^2) = 0$

$f_y = 0 \implies kx\{a^2 - x^2 - 3y^2\} = 0 \rightarrow (6)$

$(6) \implies x = 0$ or $(a^2 - x^2 - 3y^2) = 0$

$(0,0)$ [$\because$ $y = 0$ in (5) $x = 0$ in (6)

$y = 0$ & $a^2 - x^2 - 3y^2 = 0 \implies x^2 = a^2 \implies x = \pm a$

$\implies (a, 0), (-a, 0)$

$x = 0$ & $(a^2 - 3x^2 - y^2) = 0 \implies a^2 - y^2 = 0 \implies y = \pm a$

$\implies (0, a), (0, -a)$

$a^2 - 3x^2 - y^2 = 0$ & $(a^2 - x^2 - 3y^2) = 0 \Rightarrow 3x^2 + y^2 = a^2$
$x^2 + 3y^2 = a^2$

$$\begin{array}{r} 9x^2 + 3y^2 = 3a^2 \\ x^2 + 3y^2 = a^2 \\ \hline 8x^2 \quad = 2a^2 \end{array} \Longrightarrow x^2 = \frac{a^2}{4} \Rightarrow x = \pm\frac{a}{2}$$

$$x^2 = \frac{a^2}{4} \Longrightarrow \frac{3a^2}{4} + y^2 = a^2 \Longrightarrow y^2 = \frac{a^2}{4} \Rightarrow y = \pm\frac{a}{2}$$

$$\left(\frac{a}{2}, \frac{a}{2}\right), \left(\frac{a}{2}, -\frac{a}{2}\right), \left(-\frac{a}{2}, \frac{a}{2}\right), \left(-\frac{a}{2}, -\frac{a}{2}\right)$$

$$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(f_x) = ky(-6x) = -6kxy$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial y}(f_x) = k(a^2 - 3x^2 - 3y^2) \qquad \left[f_x = k(a^2y - 3x^2y - y^3)\right)$$

$$t = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}(f_y) = kx(-6y) = -6kxy \qquad f_y = kx(a^2 - x^2 - 3y^2)$$

Stationary pts (a,b)	r	s	t	$rt - s^2$	Nature of (a,b)	Inference
$(0,0)$	0	ka^2	0	$-k^2a^4 < 0$	Saddle pt	
$(a,0)$	0	$-2ka^2$	0	< 0	" "	
$(-a,0)$	0	$-2ka^2$	0	< 0	" "	
$(0,a)$	0	$-2ka^2$	0	< 0	" "	
$(0,-a)$	0	$-2ka^2$	0	< 0	" "	
$(a/2, a/2)$	$-3ka^2/2$	$-ka^2/2$	$-3ka^2/2$	$2k^2a^4$	Extrema	Maxima (as $r<0$
$(a/2, -a/2)$	$3ka^2/2$	$-ka^2/2$	$3ka^2/2$	$2k^2a^4$	Extrema	Minima (as $r>0$
$(-a/2, a/2)$	$3ka^2/2$	$-ka^2/2$	$3ka^2/2$	$2k^2a^4$	Extrema	Minima (as $r>0$
$(-a/2, -a/2)$	$-3ka^2/2$	$-ka^2/2$	$3ka^2/2$	$2k^2a^4$	Extrema	Maxima (as $r<0$)

$x^2 + y^2 + z^2 = a^2$

at $\left(\frac{a}{2}, \frac{a}{2}\right)$ or $\left(-\frac{a}{2}, -\frac{a}{2}\right) \Rightarrow z^2 = a^2 - x^2 - x^2 = a^2 - \frac{a^2}{4} - \frac{a^2}{4}$

$$z^2 = \frac{a^2}{2} \implies z = \pm \frac{a}{\sqrt{2}}$$

Highest temperature is at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{\sqrt{2}}\right), \left(\frac{a}{2}, \frac{a}{2}, -\frac{a}{\sqrt{2}}\right)$
$\left(-\frac{a}{2}, -\frac{a}{2}, \frac{a}{\sqrt{2}}\right)$ $\left(-\frac{a}{2}, -\frac{a}{2}, -\frac{a}{\sqrt{2}}\right)$

$$T_{max} = k\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\pm\frac{a}{\sqrt{2}}\right)^2 = k\left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)\left(\pm\frac{a}{\sqrt{2}}\right)^2 = \frac{ka^4}{8}$$

$$f_{max}(x,y) = T_{max}(x,y,z)$$

1) Examine the function $f(x,y)=\sin x+\sin y+\sin(x+y)$, $x,y\in(0,\pi)$ for extreme values.

2) In a plane triangle find the maximum value of $\cos A\cos B\cos C$ where A, B and C are the angles of the triangle.

$$A+B+C=\pi \implies C=\pi-(A+B)$$

$$f(A,B) = -\cos(A)\cos B\cos(A+B)$$

$$f_A = -\cos(B)\left[-\sin(A)\cos(A+B) + \cos(A)\{-\sin(A+B)\}\right]$$

$$= \cos B\sin(2A+B)$$

$$f_B = \cos A\sin(A+2B)$$

$$f_A = 0 = f_B \implies \cos B = 0 \text{ or } \sin(2A+B)=0$$
$$\cos A = 0 \text{ or } \sin(A+2B)=0$$

$B=\frac{\pi}{2}$ & $A=\frac{\pi}{2}$ A, B, C are Angles in in a triangles

$\implies$ only one angle can be $\frac{\pi}{2}$

$B=\frac{\pi}{2}$ & $A+2B=\pi \implies A=0 \implies \left(0,\frac{\pi}{2}\right)$

But angle can't be zero

$A=\frac{\pi}{2}$ & $2A+B=\pi \implies B=0$

$$\left.\begin{array}{l} 2A+B=\pi \\ A+2B=\pi \end{array}\right\} \implies A=\frac{\pi}{3} \text{ and } B=\frac{\pi}{3} \implies C=\frac{\pi}{3}$$

$$r = \frac{\partial^2 f}{\partial A^2} =$$

$$s = \frac{\partial^2 f}{\partial A \partial B} =$$

$$t = \frac{\partial^2 f}{\partial B^2} =$$

$$rt - s^2 =$$