Adding (i) and (ii) we obtain

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \qquad \ldots \text{(iii)}$$

2.6 Fourier series of arbitrary period

A function $f(x)$ need not always be defined in an interval of length 2π only. When the length of the interval is other than 2π, we shall denote it by $2l$. A general interval of length $2l$ be $(c, \; c+2l)$. It is important to note that the sine and cosine functions of the form $\sin\left(\frac{\pi x}{l}\right)$ and $\cos\left(\frac{\pi x}{l}\right)$ are periodic functions of period $2l$. It is justified as below.

$$\text{Let} \quad F(x) = \sin\left(\frac{\pi x}{l}\right); \quad G(x) = \cos\left(\frac{\pi x}{l}\right)$$

$$\text{Then} \quad F(x+2l) = \sin\left[\frac{\pi}{l}(x+2l)\right] = \sin\left(\frac{\pi x}{l} + 2\pi\right)$$

$$= \sin\left(\frac{\pi x}{l}\right) = F(x)$$

Similarly $G(x+2l) = G(x)$. Thus as discussed in article 2.3 the trigonometric series is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

If $f(x)$ defined in $(c, \; c+2l)$ satisfies Dirichlet's conditions then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

is called as the ***Fourier series of arbitrary period*** $2l$

Proceeding on similar lines as in article 2.3 we can establish Euler's formulae for the Fourier coefficients in the form

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

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Working procedure for problems

- If the period of the given function is other than 2π we first equate the period to $2l$ and obtain the value l.
- We then write the appropriate Fourier series and compute a_0, a_n, b_n associated with it.
- However if $f(x)$ is defined in an interval of the form $(-l, l)$ or $(0, 2l)$ we can compute a_0, a_n, b_n using the concept of even and odd functions taking the following table into consideration.

Nature & condition of $f(x)$	a_0	a_n	b_n
Even function $f(-x) = f(x)$ or $f(2l-x) = f(x)$	$\frac{2}{l}\int_0^l f(x)\,dx$	$\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$	0
Odd function $f(-x) = -f(x)$ or $f(2l-x) = -f(x)$	0	0	$\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

WORKED EXAMPLES.

☞ **21.** *Obtain the Fourier series of $f(x) = |x|$ in $(-l, l)$.*

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

>> The period of $f(x) = l-(-l) = 2l$ and the Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(i)$$

We shall check $f(x) = |x|$ for even or odd nature.

$$f(-x) = |-x| = |x| = f(x)$$

Hence $f(x)$ is even and consequently $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx\,; \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

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$$a_0 = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{1}{l}(l^2-0) = l \quad \therefore \quad \frac{a_0}{2} = \frac{l}{2}$$

$$a_n = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx.$$ Applying Bernoulli's rule,

$$a_n = \frac{2}{l}\left[x\cdot\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)} - (1)\cdot -\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^2}\right]_0^l.$$

$$= \frac{2}{l}\,\frac{l^2}{n^2\pi^2}\left[\cos\frac{n\pi x}{l}\right]_0^l = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$

$\because \quad \sin n\pi = 0 = \sin 0$

$$\therefore \quad a_n = \frac{-2l}{n^2\pi^2}\{1-(-1)^n\}$$

Thus the required Fourier series is given by

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{-2l}{n^2\pi^2}\{1-(-1)^n\}\cos\frac{n\pi x}{l}$$

To deduce the series we shall put $x = 0 \quad \therefore \quad f(x) = 0$

$$\therefore \quad 0 = \frac{l}{2} - \frac{2l}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}\cdot 1$$

i.e., $$\frac{-l}{2} = \frac{-2l}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\} \quad \text{or} \quad \frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}\{1-(-1)^n\}$$

But $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd}\end{cases}$$

$$\therefore \quad \frac{\pi^2}{4} = \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cdot 2 \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

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22. *Obtain the Fourier series to represent*

$f(x) = x - x^2$ *in* $-1 < x < 1$

>> Here period of $f(x) = 1 - (-1) = 2 \quad \therefore 2l = 2$ or $l = 1$

The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos(n\pi x) + \sum_1^{\infty} b_n \sin(n\pi x)$$

Since the interval is $(-1, \ 1)$ let us check the given function for even or odd nature.

$f(x) = x - x^2 \therefore f(-x) = -x - (-x)^2 = -x - x^2$ which is neither equal to $f(x)$ nor equal to $-f(x)$ *i.e.,* $f(x)$ is neither even nor odd.

We shall find the Fourier coefficients by Euler's formulae.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1}$$

i.e., $a_0 = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = \frac{-2}{3} \qquad \therefore \frac{a_0}{2} = \frac{-1}{3}$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{1}(x - x^2)\cos(n\pi x)\,dx.$$ Applying Bernoulli's rule,

$$= \left[(x - x^2)\frac{\sin(n\pi x)}{n\pi} - (1 - 2x)\cdot -\frac{\cos(n\pi x)}{n^2\pi^2} + (-2)\cdot -\frac{\sin(n\pi x)}{n^3\pi^3}\right]_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[(1 - 2x)\cos(n\pi x)\right]_{-1}^{1} \qquad \because \sin n\pi = 0$$

$$= \frac{1}{n^2\pi^2}\left\{-\cos n\pi - 3\cos n\pi\right\} = \frac{-4\cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$$

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$$\therefore \quad a_n = \frac{4(-1)^{n+1}}{n^2 \pi^2}$$

$$b_n = \frac{1}{1} \int_{-1}^{1} f(x) \sin(n\pi x)\, dx = \int_{-1}^{1} (x - x^2) \sin(n\pi x)\, dx$$

$$= \left[(x - x^2) \cdot \frac{-\cos(n\pi x)}{n\pi} - (1 - 2x) \cdot \frac{-\sin(n\pi x)}{n^2 \pi^2} + (-2) \frac{\cos(n\pi x)}{n^3 \pi^3} \right]_{-1}^{1}$$

$$= \frac{-1}{n\pi} \left\{ 0 - (-2\cos n\pi) \right\} - \frac{2}{n^3 \pi^3} (\cos n\pi - \cos n\pi)$$

$$= \frac{-2}{n\pi} (-1)^n \qquad \therefore \quad b_n = \frac{2}{n\pi} (-1)^{n+1}$$

The required Fourier series is given by

$$f(x) = \frac{-1}{3} + \frac{4}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos n\pi x + \frac{2}{\pi} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$$

☞ **23.** *Draw the graph of the function*

$$f(x) = \begin{cases} \pi x \text{ in } 0 \le x \le 1 \\ \pi(2 - x) \text{ in } 1 \le x \le 2 \end{cases}$$ *and also show that the Fourier expansion of the function* $f(x)$ *is*

$$\frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots \right)$$

>> Graph of $f(x) = y$

$f(x) = \pi x$ or $y = \pi x$ is a line passing through the origin in [0, 1]

$f(x) = \pi(2 - x)$ or $y = \pi(2 - x)$ *i.e.*, $\pi x + y = 2\pi$ or

$\frac{x}{2} + \frac{y}{2\pi} = 1$ in [1, 2], which is a straight line passing through

$(2, 0)$ and $(0, 2\pi)$.

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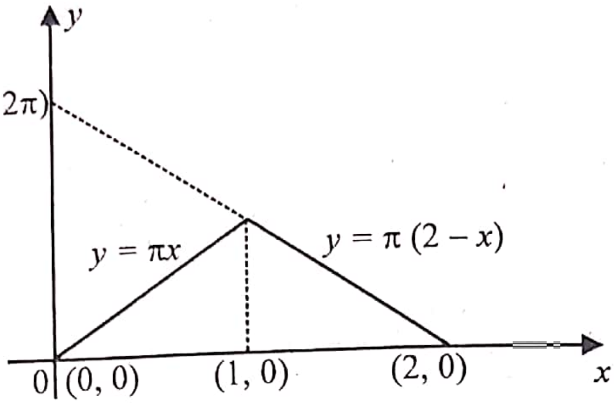

$f(x)$ is defined in $[0, 2]$ $\therefore$ Period of $f(x) = 2 - 0 = 2$

$\therefore$ $2l = 2$ or $l = 1$. The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos(n\pi x) + \sum_1^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left\{\left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2\right\}$$

$$= \pi\left\{\left(\frac{1}{2} - 0\right) + (4-2) - \left(2 - \frac{1}{2}\right)\right\} = \pi \qquad \therefore\ a_0 = \pi$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 f(x)\cos(n\pi x)\,dx + \int_1^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 \pi(2-x)\cos(n\pi x)\,dx$$

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$$= \pi \left\{ \int_0^1 x \cos(n\pi x)\, dx + \int_1^2 (2-x) \cos(n\pi x)\, dx \right\}$$

Applying Bernoulli's rule to both the integrals

$$a_n = \pi \left\{ \left[x \cdot \frac{\sin(n\pi x)}{n\pi} - 1 \cdot -\frac{\cos(n\pi x)}{n^2\pi^2} \right]_0^1 + \right.$$

$$\left. \left[(2-x) \frac{\sin(n\pi x)}{n\pi} - (-1) \cdot -\frac{\cos(n\pi x)}{n^2\pi^2} \right]_1^2 \right\}$$

$$= \frac{\pi}{n^2\pi^2} \left\{ [\cos n\pi x]_0^1 - [\cos n\pi x]_1^2 \right\} \quad \because \sin n\pi = 0 = \sin 0$$

$$= \frac{1}{n^2\pi} (\cos n\pi - \cos 0 - \cos 2n\pi + \cos n\pi).$$

But $\cos 2n\pi = 1 = \cos 0$

$$a_n = \frac{1}{n^2\pi}(-2 + 2\cos n\pi) \quad \therefore \quad a_n = \frac{-2}{\pi n^2} \{1 - (-1)^n\}$$

$$b_n = \frac{1}{1} \int_0^2 f(x) \sin(n\pi x)\, dx$$

$$= \int_0^1 f(x) \sin(n\pi x)\, dx + \int_1^2 f(x) \sin(n\pi x)\, dx$$

i.e., $$= \int_0^1 \pi x \sin(n\pi x)\, dx + \int_1^2 \pi(2-x) \sin(n\pi x)\, dx$$

$$= \pi \left\{ \left[x \cdot \frac{-\cos(n\pi x)}{n\pi} - (1) \cdot \frac{-\sin(n\pi x)}{n^2\pi^2} \right]_0^1 \right.$$

$$\left. + \left[(2-x) \cdot \frac{-\cos(n\pi x)}{n\pi} - (-1) \cdot \frac{-\sin(n\pi x)}{n^2\pi^2} \right]_1^2 \right\}$$

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$$= \frac{-\pi}{n\pi}\left\{ [x\cos(n\pi x)]_0^1 + [(2-x)\cos(n\pi x)]_1^2 \right\}$$

$$= \frac{-1}{n}\{(\cos n\pi - 0) + (0 - \cos n\pi)\} = 0$$

i.e., $b_n = 0$

The required Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(n\pi x)$$

But $1-(-1)^n = \begin{cases} 1-(+1) = 0 \text{ if } n \text{ is even} \\ 1-(-1) = 2 \text{ if } n \text{ is odd} \end{cases}$

Hence $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{n=1,3,5,\ldots}^{\infty}\frac{\cos(n\pi x)}{n^2}$ On expanding,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

Now putting $x = 0$ we have $f(x) = 0 \because f(x) = \pi x$ in [0, 1]

The Fourier series becomes

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

i.e., $-\dfrac{\pi}{2} = \dfrac{-4}{\pi}\left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots\right)$ or $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

Aliter : *(Using the concept of even and odd functions)*

$f(x)$ is defined in $(0, 2)$ which is of the form $(0, 2l)$. In the given $f(x)$ if $\phi(x) = \pi x$ and $\psi(x) = \pi(2-x)$,
$\phi(2l-x) = \phi(2-x) = \pi(2-x) = \psi(x)$

$\therefore$ $f(x)$ is even in $(0, 2)$ and hence $b_n = 0$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx, \quad a_n = \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

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$$a_0 = 2 \int_0^1 \pi x \, dx = \pi \text{ , on integration}$$

$$a_n = 2 \int_0^1 \pi x \cos(n\pi x)\, dx$$

$$= 2\pi \int_0^1 x \cos(n\pi x)\, dx = \frac{-2}{\pi n^2} \left| 1 - (-1)^n \right| \text{ , on integration.}$$

☞ **24.** *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} 1 + \dfrac{4x}{3} & in \ \dfrac{-3}{2} < x \le 0 \\ 1 - \dfrac{4x}{3} & in \ 0 \le x < \dfrac{3}{2} \end{cases}$$

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

>> $f(x)$ is defined in the interval $\left(\dfrac{-3}{2}, \dfrac{3}{2}\right)$

$\therefore$ Period of $f(x) = \dfrac{3}{2} - \left(\dfrac{-3}{2}\right) = 3$ *i.e.,* $2l = 3$ or $l = \dfrac{3}{2}$

We shall check $f(x)$ for even or odd nature.

If $\phi(x) = 1 + \dfrac{4x}{3}$, $\phi(-x) = 1 - \dfrac{4x}{3} = \psi(x)$

$\therefore$ $f(x)$ is an even function. Consequently $b_n = 0$

The Fourier series of $f(x)$ having period 3 is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos\left(\frac{n\pi x}{3/2}\right) + \sum_1^{\infty} b_n \sin\left(\frac{n\pi x}{3/2}\right)$$

i.e., $$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos\left(\frac{2n\pi x}{3}\right) + \sum_1^{\infty} b_n \sin\left(\frac{2n\pi x}{3}\right)$$

Since $f(x)$ is even we have,

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$$a_0 = \frac{2}{3/2} \int_0^{3/2} f(x)\, dx \quad \because a_0 = \frac{2}{l} \int_0^{l} f(x)\, dx$$

i.e., $$a_0 = \frac{4}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) dx$$

$$= \frac{4}{3}\left[x - \frac{2x^2}{3}\right]_0^{3/2} = \frac{4}{3}\left\{\left(\frac{3}{2} - \frac{2}{3} \cdot \frac{9}{4}\right) - 0\right\} = 0$$

$\therefore \quad a_0 = 0$

$$a_n = \frac{2}{3/2} \int_0^{3/2} f(x) \cos\left(\frac{2n\pi x}{3}\right) dx \quad \because a_n = \frac{2}{l} \int_0^{l} f(x) \cos \frac{n\pi x}{l}\, dx$$

$$= \frac{4}{3} \int_0^{3/2} \left(1 - \frac{4x}{3}\right) \cos\left(\frac{2n\pi x}{3}\right) dx.$$ Applying Bernoulli's rule

$$= \frac{4}{3}\left[\left(1 - \frac{4x}{3}\right) \frac{\sin \frac{2n\pi x}{3}}{2n\pi/3} - \left(\frac{-4}{3}\right) \cdot - \frac{\cos \frac{2n\pi x}{3}}{(2n\pi/3)^2}\right]_0^{3/2}$$

$$= \frac{4}{3} \cdot \frac{-4}{3} \cdot \frac{9}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2} = \frac{-4}{n^2\pi^2}(\cos n\pi - 1)$$

$\therefore \quad a_n = \dfrac{4}{n^2\pi^2}\left\{1 - (-1)^n\right\}$ or $a_n = \dfrac{8}{n^2\pi^2}$ where $n = 1, 3, 5, \ldots$

Hence $$f(x) = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right)$$

Putting $x = 0$ we get $f(x) = 1$. The Fourier series becomes

$$1 = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

☞ **25.** *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2$*

>> The period of $f(x) = 2 - 0 = 2 \quad \therefore \quad 2l = 2$ or $l = 1$.

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The relevant Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos n\pi x + \sum_1^\infty b_n \sin n\pi x \qquad \ldots \text{(i)}$$

$$a_0 = \int_0^2 f(x)\, dx,\ a_n = \int_0^2 f(x) \cos n\pi x\, dx,\ b_n = \int_0^2 f(x) \sin \pi x\, dx$$

(*In each of the above integrals* $1/l = 1/1 = 1$)

$$a_0 = \int_0^2 e^{-x}\, dx = \left[-e^{-x}\right]_0^2 = -(e^{-2} - 1) = 1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2}$$

$$\therefore \quad \frac{a_0}{2} = \frac{e^2 - 1}{2e^2}$$

$$a_n = \int_0^2 e^{-x} \cos n\pi x\, dx$$

We have $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$

$$\therefore \quad a_n = \left[\frac{e^{-x}}{1 + n^2\pi^2}(-\cos n\pi x + n\pi \sin n\pi x)\right]_0^2$$

$$= \frac{-1}{1 + n^2\pi^2}\left[e^{-x} \cos n\pi x\right]_0^2 \qquad \because \sin 2n\pi = 0 = \sin 0$$

$$= \frac{-1}{1 + n^2\pi^2}\left\{e^{-2} \cos 2n\pi - 1\right\} = \frac{-1}{1 + n^2\pi^2}\left(\frac{1}{e^2} - 1\right)$$

Thus $a_n = \dfrac{e^2 - 1}{e^2(1 + n^2\pi^2)}$

$$b_n = \int_0^2 e^{-x} \sin n\pi x\, dx$$

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We have $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$

$$b_n = \left[\frac{e^{-x}}{1+n^2\pi^2} (- \sin n\pi x - n\pi \cos n\pi x) \right]_0^2$$

$$= \frac{-n\pi}{1+n^2\pi^2} \left[e^{-x} \cos n\pi x \right]_0^2 = \frac{-n\pi}{1+n^2\pi^2} \left(\frac{1}{e^2} - 1 \right)$$

$$b_n = \frac{n\pi (e^2 - 1)}{e^2 (1 + n^2 \pi^2)}$$

Substituting the values of a_0, a_n, b_n in (i), the required Fourier series is given by

$$f(x) = \frac{e^2-1}{2e^2} + \sum_1^{\infty} \frac{e^2 - 1}{e^2 (1 + n^2 \pi^2)} \cos n\pi x$$

$$+ \sum_1^{\infty} \frac{n\pi (e^2 - 1)}{e^2 (1 + n^2 \pi^2)} \sin n\pi x$$

---

26. *Find the Fourier series of the periodic function defined by* $f(x) = 2x - x^2$ *in the interval* $0 < x < 3$

>> The period of $f(x) = 3 - 0 = 3$

$\therefore$ $2l = 3$ or $l = 3/2$ The Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Putting $l = \dfrac{3}{2}$ the above becomes

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos \frac{2n\pi x}{3} + \sum_1^{\infty} b_n \sin \frac{2n\pi x}{3} \qquad \ldots (i)$$

We shall find Fourier coefficients from Euler's formulae for the interval (0, 3) with reference to the Fourier series (i). That is

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$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx,\; a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx,$$

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}\,dx$$

In each of the above integrals $1/l = 1/(3/2) = 2/3$

$$a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\{(9-9)-0\} = 0 \quad \therefore \quad \frac{a_0}{2} = 0$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx. \text{ Applying Bernoulli's rule,}$$

$$a_n = \frac{2}{3}\left[(2x - x^2)\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3} - (2-2x)\cdot\frac{-\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2}\right.$$

$$\left. + (-2)\cdot\frac{-\sin\frac{2n\pi x}{3}}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\,\frac{9}{4n^2\pi^2}\left[(2-2x)\cos\frac{2n\pi x}{3}\right]_0^3$$

The first and third terms vanish $\because \sin 2n\pi = 0 = \sin 0$

$$= \frac{3}{2n^2\pi^2}\{(2-6)\cos 2n\pi - (2-0)\cos 0\}$$

But $\quad \cos 2n\pi = 1 = \cos 0$

$$\therefore \quad a_n = \frac{3}{2n^2\pi^2}(-6) = \frac{-9}{n^2\pi^2} \qquad \therefore \quad a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx. \text{ Applying Bernoulli's rule,}$$

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$$b_n = \frac{2}{3}\left[(2x - x^2) \cdot \frac{-\cos\dfrac{2n\pi x}{3}}{2n\pi/3} - (2-2x) \cdot \frac{-\sin\dfrac{2n\pi x}{3}}{(2n\pi/3)^2} + (-2) \cdot \frac{\cos\dfrac{2n\pi x}{3}}{(2n\pi/3)^3} \right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\left\{ (2x - x^2) \cos\frac{2n\pi x}{3} \right\} - \frac{54}{8n^3\pi^3}\left\{ \cos\frac{2n\pi x}{3} \right\} \right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\left\{ (6-9)\cos 2n\pi - 0 \right\} - \frac{54}{8n^3\pi^3}\left\{ \cos 2n\pi - \cos 0 \right\} \right]_0^3$$

$$= \frac{2}{3}\left\{ \frac{-3}{2n\pi}(-3) \right\} \quad \therefore \quad b_n = \frac{3}{n\pi}$$

Substituting the values of a_0, a_n, b_n in (i) the required Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\frac{2n\pi x}{3}$$

☞ **27** *Obtain the Fourier series for the function* $f(x) = 2x - x^2$ in $0 \le x \le 2$

>> Comparing the given interval $(0, 2)$ with $(0, 2l)$ we have $2l = 2$ or $l = 1$. By data $f(x) = x(2-x)$

$f(2l - x) = f(2-x) = (2-x)(2-\overline{2-x}) = (2-x)x = f(x)$

$\therefore$ $f(x)$ is even in $(0, 2)$ and hence $b_n = 0$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx, \quad a_n = \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$a_0 = 2\int_0^1 (2x - x^2)\,dx = 2\left[x^2 - \frac{x^3}{3} \right]_0^1 = 2\left[\left(1 - \frac{1}{3} \right) - 0 \right] = \frac{4}{3}$$

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Thus $a_0 = 4/3$ and $a_0/2 = 2/3$

$$a_n = 2 \int_0^1 (2x - x^2)\cos(n\pi x)\,dx$$. Applying Bernoulli's rule,

$$a_n = 2\left[(2x - x^2)\frac{\sin(n\pi x)}{n\pi} - (2-2x)\cdot\frac{-\cos(n\pi x)}{n^2\pi^2} + (-2)\cdot\frac{-\sin(n\pi x)}{n^3\pi^3}\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[(2-2x)\cos(n\pi x)\right]_0^1 = \frac{2}{n^2\pi^2}\left[0 - 2\cos 0\right] = \frac{-4}{n^2\pi^2}$$

The Fourier series of period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\therefore \quad \boldsymbol{f(x) = 2x - x^2 = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2}\cos(n\pi x)}$$

☞ **28.** *Obtain the Fourier series of the saw-tooth function $f(t) = Et/T$ for $0 < t < T$ given that $f(t+T) = f(t)$ for all $t > 0$*

>> We have $2l = T$ or $l = T/2$ and the associated Fourier series of period $2l = T$ is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{T/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T/2}\right)$$

i.e., $$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + \sum_{1}^{\infty} b_n \sin\left(\frac{2n\pi t}{T}\right)$$

We compute Fourier coefficients by Euler's formulae.

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$$a_0 = \frac{1}{T/2} \int_0^T f(t)\, dt\,,\quad a_n = \frac{1}{T/2} \int_0^T f(t) \cos\left(\frac{2n\pi t}{T}\right) dt$$

$$b_n = \frac{1}{T/2} \int_0^T f(t) \sin\left(\frac{2n\pi t}{T}\right) dt$$

$$a_0 = \frac{2}{T} \int_0^T \frac{Et}{T}\, dt = \frac{2E}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{E}{T^2}(T^2 - 0) = E$$

$$a_n = \frac{2}{T} \int_0^T \frac{Et}{T} \cos\frac{2n\pi t}{T}\, dt = \frac{2E}{T^2} \int_0^T t \cos\frac{2n\pi t}{T}\, dt$$

i.e., $$= \frac{2E}{T^2}\left[t \cdot \frac{\sin\frac{2n\pi t}{T}}{(2n\pi/T)} - 1 \cdot \frac{-\cos\frac{2n\pi t}{T}}{(2n\pi/T)^2} \right]_0^T$$

$$= \frac{2E}{T^2} \quad \frac{T^2}{4n^2\pi^2}(\cos 2n\pi - \cos 0) = 0 \qquad \therefore \quad a_n = 0$$

$$b_n = \frac{2}{T} \cdot \frac{E}{T} \int_0^T t \sin\frac{2n\pi t}{T}\, dt$$

$$= \frac{2E}{T^2}\left[t \cdot \frac{-\cos\frac{2n\pi t}{T}}{(2n\pi/T)} - 1 \cdot \frac{-\sin\frac{2n\pi t}{T}}{(2n\pi/T)^2} \right]_0^T$$

$$= \frac{-E}{n\pi T}\left[t \cos\frac{2n\pi t}{T} \right]_0^T = \frac{-E}{n\pi T}(T \cos 2n\pi - 0) = \frac{-E}{n\pi}$$

Thus $b_n = -E/n\pi$

The required Fourier series is given by

$$\boldsymbol{f(t) = \frac{E}{2} - \frac{E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{2n\pi t}{T}}$$

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☞ **29.** *If* $f(x) = \begin{cases} 2-x & in\ 0 \le x \le 4 \\ x-6 & in\ 4 \le x \le 8 \end{cases}$

Express $f(x)$ *as a Fourier series and hence deduce that*

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

>> Comparing the given interval $(0, 8)$ with $(0, 2l)$ we have $2l = 8$ or $l = 4$. The Fourier series having period 8 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

In the given $f(x)$ let $\phi(x) = 2-x$, $\psi(x) = x-6$

$\therefore \quad \phi(2l-x) = \phi(8-x) = 2-(8-x) = x-6 = \psi(x)$

Thus $f(x)$ is even in $(0, 8)$ and hence $b_n = 0$

$$a_0 = \frac{2}{4}\int_0^4 f(x)\,dx, \quad a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$a_0 = \frac{1}{2}\int_0^4 (2-x)\,dx = \frac{1}{2}\left[2x - \frac{x^2}{2}\right]_0^4 = \frac{1}{2}[(8-8)-0] = 0$$

$$a_n = \frac{1}{2}\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,dx$$ Applying Bernoulli's rule,

$$= \frac{1}{2}\left[\frac{(2-x)\sin\frac{n\pi x}{4}}{(n\pi/4)} - (-1)\cdot\frac{-\cos\frac{n\pi x}{4}}{(n\pi/4)^2}\right]_0^4$$

$$= \frac{-8}{n^2\pi^2}\left[\cos\frac{n\pi x}{4}\right]_0^4 = \frac{-8}{n^2\pi^2}\{\cos n\pi - 1\}$$

$$\therefore \quad a_n = \frac{8}{n^2\pi^2}\{1-(-1)^n\} = \frac{16}{n^2\pi^2} \quad \text{where } n = 1, 3, 5, \ldots\ldots$$

Hence the required Fourier series is given by

$$f(x) = \sum_{n=1,3,5\cdots}^{\infty} \frac{16}{n^2\pi^2} \cos\left(\frac{n\pi x}{4}\right)$$

To deduce the series we put $x = 0$ $\therefore$ $f(x) = 2-0 = 2$ and hence the Fourier series becomes

$$2 = \frac{16}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2} \cdot 1 \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Equivalently we have $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

☞ **30.** *Find the Fourier expansion of the function $f(x)$ defined by*

$$f(x) = \begin{cases} 0 & \text{in } -2 < x < -1 \\ 2 & \text{in } -1 < x < 1 \\ 0 & \text{in } \quad 1 < x < 2 \end{cases}$$

>> $f(x)$ is defined in $(-2,\ 2)$ $\therefore$ period of $f(x) = 2-(-2) = 4$

i.e., $2l = 4$ or $l = 2$. The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \qquad \ldots \text{(i)}$$

The given $f(x)$ can be written as below.

Interval of x	$(-2, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 2)$
$f(x)$	0	2	2	0

i.e., $f(x) = \begin{cases} \phi(x) & \text{in } (-2,\ 0) \\ \psi(x) & \text{in } (0,\ 2) \end{cases}$

where $\phi(x) = 0$ or 2, $\psi(x) = 2$ or 0. Obviously $\phi(-x) = \psi(x)$

$\therefore$ $f(x)$ is even and consequently $b_n = 0$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx, \quad a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

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i.e., $$a_0 = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 2\,dx + 0 = [\,2x\,]_0^1 = 2$$

$$a_n = \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 2\cos\frac{n\pi x}{2}\,dx + 0$$

$$= 2\left[\frac{\sin\frac{n\pi x}{2}}{(n\pi/2)}\right]_0^1 = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Thus the required Fourier series is given by

$$f(x) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

EXERCISES

Find the Fourier series of the following functions over the indicated intervals

1. $f(x) = -1 + x;\ -\pi < x < \pi$

2. $f(x) = x^2;\ 0 < x < 2\pi$

3. $f(x) = \pi^2 - x^2;\ -\pi \le x \le \pi$. Deduce that

   (a) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \cdots = \frac{\pi^2}{12}$

   (b) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

4. $f(x) = \frac{x}{12}(\pi^2 - x^2);\ -\pi < x < \pi$

5. $f(x) = x \sin x;\ (0, 2\pi)$. Deduce that

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12. Show that the cosine half range series for $f(x) = a \sin x$ in $(0, \pi)$ is $\frac{4a}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{1 \cdot 3} - \frac{\cos 4x}{3 \cdot 5} - \frac{\cos 6x}{5 \cdot 7} \cdots\right)$ and hence deduce that $\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \cdots = \frac{\pi - 2}{4}$

ANSWERS

1. $\frac{1}{3} + \frac{4}{\pi^2} \sum_1^\infty \frac{1}{n^2} \cos \frac{n\pi x}{l}$

2. $1 - \frac{8}{\pi^2} \sum_1^\infty \left[1 - (-1)^n\right] \cos \frac{n\pi x}{2}$

3. $\frac{\sin a\pi}{a\pi} + \frac{2a}{\pi} \sum_1^\infty (-1)^{n+1} \frac{\sin a\pi \cos nx}{n^2 - a^2}$

4. $\frac{2}{\pi^2} \sum_1^\infty \frac{1}{n^2} \left\{1 + \cos n\pi - 2 \cos \frac{n\pi}{2}\right\} \cos n\pi x$

5. $\frac{-16}{\pi^2} \sum_1^\infty \frac{1}{n^2} \left\{1 + \cos n\pi - 2 \cos \frac{n\pi x}{8}\right\}$

6. $\frac{-4}{\pi} \sum_{n = 2, 4, 6, \ldots}^\infty \frac{1}{n} \sin n\pi x$

7. $\frac{-8}{\pi^3} \sum_{n = 1, 3, 5, \ldots}^\infty \frac{1}{n^3} \sin 2n\pi x$

8. $\frac{2}{\pi} \sum_1^\infty \frac{1}{n} \sin nx$

9. $\frac{4}{\pi^2} \sum_1^\infty \frac{1}{n^2} \sin \frac{n\pi}{2} \sin n\pi x$

10. $\frac{4kl}{\pi^2} \sum_1^\infty \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l}$

✠ 2.8 Harmonic Analysis

So far, we have discussed the methods of obtaining the Fourier series of a known function $f(x)$ in a given interval. However, there will also be situations where in there will be no known functional expression for $f(x)$ but only the values at some equidistant points will be known.

Harmonic analysis is the process of finding the constant term and the first few cosine and sine terms numerically.

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The Fourier series of period 2π of a function $y = f(x)$ will be of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$, where the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx\,,\quad a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx,$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx\, dx.$$

$a_0/2$ is called the constant term and the groups of terms $(a_1 \cos x + b_1 \sin x)$, $(a_2 \cos 2x + b_2 \sin 2x)$ etc. are called the *first harmonics, second harmonics* etc.,

To derive the relevant formulae, we need the following principle.
The mean value of a continuous function $y = f(x)$ in the range (a, b) is given by $\frac{1}{b-a} \int_a^b f(x)\, dx$

Suppose we have a set of N values of $y = f(x)$ having period 2π at equidistant points of x in the interval $c \le x < c + 2\pi$ or $c < x \le c + 2\pi$. [*if the values of y at $x = c$ and $x = c + 2\pi$ are given we must omit one of them. Infact* $(y)_{x=c} = (y)_{x=c+2\pi}$ *by the periodic property* $f(x) = f(x + 2\pi)$] the Fourier coefficients a_0, a_n, b_n, assume the following form by the above stated principle.

$$a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx = 2 \left[\frac{1}{(c+2\pi) - c} \int_c^{c+2\pi} f(x)\, dx \right]$$

Here $a = c, b = c + 2\pi$

ie., $a_0 = 2$ [mean value of $f(x) = y$ in $(c,\ c + 2\pi)$]

or $a_0 = 2 \left[\frac{\Sigma y}{N} \right]$ *i.e.,* $a_0 = \frac{2}{N} \Sigma y$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx = 2\left[\frac{1}{(c+2\pi)-c}\int_c^{c+2\pi} f(x)\cos nx\,dx\right]$$

$= 2$[mean value of $y\cos nx$ in $(c,\ c+2\pi)$]

or $\quad a_n = 2\left[\frac{\Sigma y\cos nx}{N}\right]$ *i.e.,* $a_n = \frac{2}{N}\Sigma\, y\cos nx$

Similarly $b_n = 2\left[\frac{\Sigma y\sin nx}{N}\right]$ *i.e.,* $b_n = \frac{2}{N}\Sigma\, y\sin nx$

Suppose we have a set of N values of $y = f(x)$ having period $2l$ at equidistant points of x in the interval $c \le x < c+2l$ or $c < x \le c+2l$

$$a_n = 2\left[\Sigma\frac{y\cos(n\pi x/l)}{N}\right];\quad b_n = 2\left[\Sigma\frac{y\sin(n\pi x/l)}{N}\right].$$

Taking $\theta = \frac{\pi x}{l}$

$$a_n = \frac{2}{N}\Sigma\, y\cos n\theta,\quad b_n = \frac{2}{N}\Sigma\, y\sin n\theta.$$

Note : *All these formulae holds good for halfrange Fourier series also.*

Working procedure for problems

We have to first write down the period of $y = f(x)$ from the given range of the values of x. If the period is 2π, depending on the harmonics required we prepare the relevant table along with the summations (Σ) of y; $y\cos x$, $y\cos 2x$...; , $y\sin x$, $y\sin 2x$... and compute the harmonics using the formulae derived by taking $n = 1,\ 2,\ \ldots$. If the period is not 2π we equate it with $2l$ to obtain the value of l. The summations of y; $y\cos\theta$, $y\cos 2\theta$,...; $y\sin\theta$, $y\sin 2\theta$...where $\theta = \pi x/l$ will be required to compute the desired harmonics.

WORKED EXAMPLES

☞ **43.** *Determine the constant term and the first cosine and sine terms of the Fourier series expansion of y from the following data.*

x^0	0	45	90	135	180	225	270	315
y	2	3/2	1	1/2	0	1/2	1	3/2

>> Here the interval of x is 0^0 to 360^0 *i.e.,* $0 \le x < 2\pi$ .We have to find a_0; $(a_1,\ b_1)$. This requires the summation of y, $y\cos x$, $y\sin x$.

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	2.0	1	2.0	0	0
45	1.5	0.7071	1.06065	0.7071	1.06065
90	1.0	0	0	1	1.0
135	0.5	– 0.7071	– 0.35355	0.7071	0.35355
180	0	– 1	0	0	0
225	0.5	– 0.7071	– 0.35355	– 0.7071	– 0.35355
270	1.0	0	0	– 1	– 1
315	1.5	0.7071	1.06065	– 0.7071	– 1.06065
Totals	**8.0**		**3.4142**		**0**

$$a_0 = \frac{2}{N}\Sigma y,\ a_1 = \frac{2}{N}\Sigma y\cos x,\ b_1 = \frac{2}{N}\Sigma y\sin x,\ N = 8.$$

Also from the table,

$$\Sigma y = 8.0,\ \Sigma y\cos x = 3.4142,\ \Sigma y\sin x = 0.$$

$$\therefore \quad a_0 = \frac{2}{8}(8.0) = 2 \quad \text{or} \quad \frac{a_0}{2} = 1$$

$$a_1 = \frac{2}{8}(3.4142) = 0.85355$$

$$b_1 = \frac{2}{8}(0) = 0$$

Thus the Fourier series of y up to the first harmonic is

$$y = a_0/2 + a_1\cos x + b_1 \sin x$$

i.e., $\mathbf{y = 1 + 0.85355 \cos x}$

---

44. *Obtain the Fourier series of y up to the second harmonics for the following values.*

x^0	45	90	135	180	225	270	315	360
y	4.0	3.8	2.4	2.0	– 1.5	0	2.8	3.4

>> The interval of x is $0 < x \le 2\pi$ and period of $y = f(x)$ is 2π

We have to compute a_0; a_1, b_1; a_2, b_2

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x^0	y	$\cos x$	$y\cos x$	$\sin x$	$y\sin x$
45	4.0	0.7071	2.8284	0.7071	2.8284
90	3.8	0	0	1	3.8
135	2.4	– 0.7071	– 1.69704	0.7071	1.69704
180	2.0	– 1	– 2.0	0	0
225	– 1.5	– 0.7071	1.06065	– 0.7071	1.06065
270	0	0	0	– 1	0
315	2.8	0.7071	1.97988	– 0.7071	– 1.97988
360	3.4	1	3.4	0	0
Totals	**16.9**		**5.57189**		**7.40621**

$\cos 2x$	$y\cos 2x$	$\sin 2x$	$y\sin 2x$
0	0	1	4.0
– 1	– 3.8	0	0
0	0	– 1	– 2.4
1	2.0	0	0
0	0	1	– 1.5
– 1	0	0	0
0	0	– 1	– 2.8
1	3.4	0	0
Totals	**1.6**		**– 2.7**

$$a_0 = \frac{2}{N}\Sigma y,\ a_1 = \frac{2}{N}\Sigma y\cos x,\ b_1 = \frac{2}{N}\Sigma y\sin x$$

$$a_2 = \frac{2}{N}\Sigma y\cos 2x,\ b_2 = \frac{2}{N}\Sigma y\sin 2x \quad \text{Here } N = 8,\ \therefore\ \frac{2}{N} = \frac{1}{4}$$

Also from the table,

$$\Sigma y = 16.9,\ \Sigma y\cos x = 5.57189,\ \Sigma y\sin x = 7.40621,$$
$$\Sigma y\cos 2x = 1.6,\ \Sigma y\sin 2x = -2.7$$

Thus $a_0 = \frac{1}{4}(16.9) = 4.225,\quad \frac{a_0}{2} = 2.1125,$

$$a_1 = \frac{1}{4}(5.57189) = 1.393. \qquad b_1 = \frac{1}{4}(7.40621) = 1.8516$$

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$$a_2 = \frac{1}{4}(1.6) = 0.4 \qquad b_2 = \frac{1}{4}(-2.7) = -0.675$$

The Fourier series of y upto the second harmonic is given by

$$y = a_0/2 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$

i.e., $\mathbf{y = 2.1125 + (1.393 \cos x + 1.8516 \sin x) + (0.4 \cos 2x - 0.675 \sin 2x)}$

☞ **45.** *Given the following table*

x^0	0	60^0	120^0	180^0	240^0	300^0
y	7.9	7.2	3.6	0.5	0.9	6.8

Obtain the Fourier series neglecting terms higher than first harmonics.

>> Here the interval of x is $(0^0, 360^0)$ *i.e.,* $0 \le x < 2\pi$. We are required to find a_0, a_1, b_1 only.

x^0	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$
0	7.9	1	7.9	0	0
60	7.2	0.5	3.6	0.866	6.2352
120	3.6	– 0.5	– 1.8	0.866	3.1176
180	0.5	– 1	– 0.5	0	0
240	0.9	– 0.5	– 0.45	– 0.866	– 0.7794
300	6.8	0.5	3.4	– 0.866	– 5.8888
Totals	**26.9**		**12.15**		**2.6846**

Here $N = 6$ $\quad \therefore \frac{2}{N} = \frac{1}{3}$

$$a_0 = \frac{2}{N}\Sigma y = \frac{1}{3}(26.9) = 8.9667 \quad \therefore \frac{a_0}{2} = 4.48335$$

$$a_1 = \frac{2}{N}\Sigma y \cos x = \frac{1}{3}(12.15) = 4.05$$

$$b_1 = \frac{2}{N}\Sigma y \sin x = \frac{1}{3}(2.6846) = 0.8949$$

The Fourier series upto the first harmonic is

$$y = a_0/2 + (a_1 \cos x + b_1 \sin x)$$

i.e., $\mathbf{y = 4.48335 + (4.05 \cos x + 0.8949 \sin x)}$

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☞ **46.** *The turning moment T on the crank shaft of a steam engine for the crank angle θ is given as follows.*

θ^0	0	30	60	90	120	150	180	210	240	270	300	330
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

Expand T as a Fourier series up to first harmonics.

>> Here the interval of θ is $0 \le \theta < 2\pi$. Period of T is 2π. We are required to find a_0, a_1, b_1. The corresponding formulae are

$$a_0 = \frac{2}{N}\Sigma T,\ a_1 = \frac{2}{N}\Sigma T\cos\theta,\ b_1 = \frac{2}{N}\Sigma T\sin\theta.\ N = 12,\ \frac{2}{N} = \frac{1}{6}$$

θ^0	T	$\cos\theta$	$T\cos\theta$	$\sin\theta$	$T\sin\theta$
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	− 0.5	− 4.05	0.866	7.0146
150	8.3	− 0.866	− 7.1878	0.5	4.15
180	7.9	− 1	− 7.9	0	0
210	6.8	− 0.866	− 5.8888	− 0.5	− 3.4
240	5.5	− 0.5	− 2.75	− 0.866	− 4.763
270	4.1	0	0	− 1	− 4.1
300	2.6	0.5	1.3	− 0.866	− 2.2516
330	1.2	0.866	1.0392	− 0.5	− 0.6
Totals	**59.4**		**− 20.4992**		**8.9032**

$$a_0 = \frac{1}{6}\,\Sigma T \quad = \frac{1}{6}\,(59.4) = 9.9 \quad \therefore \quad \frac{a_0}{2} = 4.95$$

$$a_1 = \frac{1}{6}\,\Sigma T\cos\theta = \frac{1}{6}\,(-20.4992) - 3.4165$$

$$b_1 = \frac{1}{6}\,\Sigma T\sin\theta = \frac{1}{6}\,(8.9032) = 1.4839$$

The required Fourier series is given by

$$T = f(\theta) = a_0/2 + (a_1\cos\theta + b_1\sin\theta)$$

ie., $\quad$ **$T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta$**

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47. *Express y as a Fourier series upto the third harmonics given*

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

>> Here the interval of x is $(0 \le x \le 2\pi)$ and the values of y at $x = 0$ and $x = 2\pi$ must be same by the periodic property $f(x+2\pi) = f(x)$. In the given problem the values of y at $x = 0$ and 2π both are given and we must omit one of them. Let us omit the last value. The values of x in degrees are 0, 60, 120, 180, 240, 300 and $N = 6$.

The relevant table is formulated by considering the values of $\cos x$, $\sin x$; $\cos 2x$, $\sin 2x$; $\cos 3x$, $\sin 3x$ rounded off to three places of decimals and then multiplied by the values of y.

0	y	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.98	1.98	1.98	1.98	0	0	0
60	1.3	0.65	-0.65	-1.3	1.1258	1.1258	0
120	1.05	-0.525	-0.525	1.05	0.9093	-0.9093	0
180	1.3	-1.3	1.3	-1.3	0	0	0
240	-0.88	0.44	0.44	-0.88	0.76208	-0.76208	0
300	-0.25	-0.125	0.125	0.25	0.2165	0.2165	0
Totals	**4.5**	**1.12**	**2.67**	**-0.2**	**3.01368**	**-0.32908**	**0**

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$$a_0 = \frac{2}{N}\Sigma y \qquad = \frac{1}{3}(4.5) = 1.5 \quad \therefore \frac{a_0}{2} = 0.75$$

$$a_1 = \frac{2}{N}\Sigma y\cos x \quad = \frac{1}{3}(1.12) = 0.3733$$

$$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{1}{3}(2.67) = 0.89$$

$$a_3 = \frac{2}{N}\Sigma y\cos 3x = \frac{1}{3}(-0.2) = -0.0667$$

$$b_1 = \frac{2}{N}\Sigma y\sin x \quad = \frac{1}{3}(3.01368) = 1.00456$$

$$b_2 = \frac{2}{N}\Sigma y\sin 2x \quad = \frac{1}{3}(-0.32908) = -0.1097$$

$$b_3 = \frac{2}{N}\Sigma y\sin 3x \quad = \frac{1}{3}(0) = 0.$$

Fourier series up to third harmonics is given by

$$y = a_0/2 + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + (a_3\cos 3x + b_3\sin 3x)$$

i.e., $\mathbf{y = 0.75 + (0.3733\ \cos x + 1.00456\sin x) + (0.89\cos 2x - 0.1097\sin 2x) + (-0.0667\cos 3x)}$

☞ **48.** *Find the Fourier series to represent $y(x)$ upto the second harmonic from the follwing data :*

x^0	30	60	90	120	150	180	210	240	270	300	330	360
y	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

>> The period of $y(x)$ is $2\pi = 360^0$ and the Fourier series upto the second harmonic is given by

$$y(x) = a_0/2 + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$$

where $a_0 = \frac{2}{N}\Sigma y$, $a_1 = \frac{2}{N}\Sigma y\cos x$, $a_2 = \frac{2}{N}\Sigma y\cos 2x$

$b_1 = \frac{2}{N}\Sigma y\sin x$, $b_2 = \frac{2}{N}\Sigma y\sin 2x$; Here $N = 12$ and

and we prepare the following table

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x^0	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
30	2.34	0.87	0.5	0.5	0.87	2.0358	1.17	1.17	2.0358
60	3.01	0.5	− 0.5	0.87	0.87	1.505	− 1.505	2.6187	2.6187
90	3.68	0	− 1	1	0	0	− 3.68	3.68	0
120	4.15	− 0.5	− 0.5	0.87	− 0.87	− 2.075	− 2.075	3.6105	− 3.6105
150	3.69	− 0.87	0.5	0.5	− 0.87	− 3.2103	1.845	1.845	− 3.2103
180	2.20	− 1	1	0	0	− 2.2	2.2	0	0
210	0.83	− 0.87	0.5	− 0.5	0.87	− 0.7221	0.415	− 0.415	0.7221
240	0.51	− 0.5	− 0.5	− 0.87	0.87	− 0.255	-0.255	-0.4437	0.4437
270	0.88	0	− 1	− 1	0	0	− 0.88	− 0.88	0
300	1.09	0.5	− 0.5	− 0.87	− 0.87	0.545	− 0.545	− 0.9483	− 0.9483
330	1.19	0.87	0.5	− 0.5	− 0.87	1.0353	0.595	− 0.595	− 1.0353
360	1.64	1	1	0	0	1.64	1.64	0	0
Totals	25.21					− 1.7013	− 1.075	9.6422	− 2.9841

$$a_0 = \frac{25.21}{6} = 4.2017, \quad \frac{a_0}{2} = 2.1$$

$$a_1 = \frac{-1.7013}{6} = -0.28, \qquad a_2 = \frac{-1.075}{6} = -0.18$$

$$b_1 = \frac{9.6422}{6} = 1.61, \qquad b_2 = \frac{-2.9841}{6} = -0.5$$

Thus the required Fourier series upto the second harmonic is given by

$$\mathbf{y(x) = 2.1 + (-0.28 \cos x + 1.61 \sin x)}$$
$$\mathbf{+ (-0.18 \cos 2x - 0.5 \sin 2x)}$$

☞ **49.** *Compute the first two harmonics of the Fourier series of $f(x)$ given the following table*

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
$f(x)$	1.0	1.4	1.9	1.7	1.5	1.2	1.0

>> We have values of $f(x) = y$ in the interval $0 \le x \le 2\pi$ and hence omit the last value $f(2\pi) = 1.0$ which is same as $f(0)$. The relevant table for computing the first two harmonics is as follows.

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x^0	y	$\cos x$	$\cos 2x$	$\sin x$	$\sin 2x$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1	1	1	0	0	1	1	0	0
60	1.4	0.5	− 0.5	0.866	0.866	0.7	− 0.7	1.2124	1.2124
120	1.9	− 0.5	− 0.5	0.866	− 0.866	− 0.95	− 0.95	1.6454	− 1.6454
180	1.7	− 1	1	0	0	− 1.7	1.7	0	0
240	1.5	− 0.5	− 0.5	− 0.866	0.866	− 0.75	− 0.75	1.299	1.299
300	1.2	0.5	− 0.5	− 0.866	− 0.866	0.6	− 0.6	− 1.0392	− 1.0392
Totals						− 1.1	− 0.3	3.1176	− 0.1732

$$a_1 = \frac{2}{N} \Sigma\, y \cos x = \frac{2}{6}(-1.1) = -0.367$$

$$a_2 = \frac{2}{N} \Sigma\, y \cos 2x = \frac{2}{6}(-0.3) = -0.1$$

$$b_1 = \frac{2}{N} \Sigma\, y \sin x = \frac{2}{6}(3.1176) = 1.0392$$

$$b_2 = \frac{2}{N} \Sigma\, y \sin 2x = \frac{2}{6}(-0.1732) = -0.0577$$

The first two harmonics are

$$(a_1 \cos x + b_1 \sin x) \text{ and } (a_2 \cos 2x + b_2 \sin 2x). \text{ That is}$$

$$\mathbf{(-0.367 \cos x + 1.0392 \sin x) \text{ and } (-0.1 \cos 2x - 0.0577 \sin 2x)}$$

☞ **50.** *Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the table*

x	0	1	2	3	4	5
y	9	18	24	28	26	20

>> The values at 0, 1, 2, 3, 4, 5 are given $(N = 6)$ and hence the interval of x should be $0 \le x < 6$ *i.e.,* $(0, 6)$

$\therefore$ length of the interval is $6 - 0 = 6$. Comparing with $2l$ we have $2l = 6$. $\therefore$ $l = 3$. The Fourier series of period $2l$ is

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Since $l = 3$, the series containing the first harmonics is

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$$y = f(x) = \frac{a_0}{2} + a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3}$$

Writing $\frac{\pi x}{3} = \theta,\ y = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta\ ;\ N = 6 \quad \therefore \quad \frac{2}{N} = \frac{1}{3}$

x	$\theta = \pi x/3$	y	$\cos\theta$	$y\cos\theta$	$\sin\theta$	$y\sin\theta$
0	0	9	1	9	0	0
1	60^0	18	0.5	9	0.866	15.588
2	120^0	24	-0.5	-12	0.866	20.784
3	180^0	28	-1	-28	0	0
4	240^0	26	-0.5	-13	-0.866	-22.516
5	300^0	20	0.5	10	-0.866	-17.32
Total		**125**		**−25**		**−3.464**

$$\therefore \quad a_0 = \frac{2}{N}\Sigma y = \frac{1}{3}(125) \approx 41.67 \quad \therefore \quad \frac{a_0}{2} = 20.835$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{1}{3}(-25) \approx -8.333$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{1}{3}(-3.464) \approx -1.155$$

Constant term $= a_0/2 = \mathbf{20.835}$

Coefficient of the first cosine term $= a_1 = \mathbf{-8.333}$

Coefficient of the first sine term $= b_1 = \mathbf{-1.155}$

☞ **51.** *Express y as a Fourier series up to the third harmonics given the following values.*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

>> As in the previous problem the interval of definition is $(0, 6)$

$$\therefore \quad 2l = 6 \text{ i.e., } l = 3, N = 6, \quad \therefore \quad \frac{2}{N} = \frac{1}{3}$$

Fourier series up to the third harmonics is

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$$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l}\right) + \left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l}\right)$$

$$+ \left(a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l}\right) \quad \text{Here } l = 3$$

$$\therefore \quad y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right)$$

$$+ \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

Putting $\pi x/3 = \theta$

$$y = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

$$+ (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

x	$\theta = \pi x/3$	y	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	4	4	4	4
1	60^0	8	4	– 4	– 8
2	120^0	15	– 7.5	– 7.5	15
3	180^0	7	– 7	7	– 7
4	240^0	6	– 3	– 3	6
5	300^0	2	1	– 1	– 2
Total		**42**	**– 8.5**	**– 4.5**	**8**

$y\sin\theta$	$y\sin 2\theta$	$y\sin 3\theta$
0	0	0
6.928	6.928	0
12.99	– 12.99	0
0	0	0
– 5.196	5.196	0
– 1.732	– 1.732	0
12.99	**– 2.598**	**0**

$$a_0 = \frac{2}{N}\Sigma y \qquad = \frac{1}{3}(42) = 14 \quad \therefore \quad \frac{a_0}{2} = 7$$

$$a_1 = \frac{2}{N}\Sigma y \cos\theta \quad = \frac{1}{3}(-8.5) = -2.833$$

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$$b_1 = \frac{2}{N}\Sigma y \sin\theta = \frac{1}{3}(12.99) = 4.33$$

$$a_2 = \frac{2}{N}\Sigma y \cos 2\theta = \frac{1}{3}(-4.5) = -1.5$$

$$b_2 = \frac{2}{N}\Sigma y \sin 2\theta = \frac{1}{3}(-2.598) = -0.866$$

$$a_3 = \frac{2}{N}\Sigma y \cos 3\theta = \frac{1}{3}(8) \approx 2.667$$

$$b_3 = \frac{2}{n}\Sigma y \sin 3\theta = \frac{1}{3}(0) = 0$$

Hence the required Fourier series upto the third harmonics is

$$\mathbf{y = 7 - 2.833 \cos\frac{\pi x}{3} + 4.33 \sin\frac{\pi x}{3}}$$
$$\mathbf{- 1.5 \cos\frac{2\pi x}{3} - 0.866 \sin\left(\frac{2\pi x}{3}\right) + 2.667 \cos\pi x}$$

☞ **52.** *Obtain the constant term and the first three coefficients in the Fourier cosine series for y using the following table*

x	0	1	2	3	4	5
y	4	8	15	7	6	2

>> Here the interval of x is $(0, 6)$ *i.e.,* $0 \le x < 6$ and since the coefficients of the Fourier cosine series are to be found we have to conclude that it should be the cosine half range Fourier series of $y = f(x)$ in $(0, 6)$. Comparing with half the range $(0, l)$ we get $l = 6$ and the

Fourier cosine series is of the form $y = \dfrac{a_0}{2} + \sum\limits_{1}^{\infty} a_n \cos\dfrac{n\pi x}{l}$

i.e., $$y = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{6}\right) + a_2 \cos\left(\frac{2\pi x}{6}\right) + a_3 \cos\left(\frac{3\pi x}{6}\right) + \ldots$$

Putting $\pi x/6 = \theta$ the Fourier series upto third harmonics assumes the form

$$y = a_0/2 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta \qquad \ldots(i)$$

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We have to compute $a_0/2$ (*cosntant term*), a_1, a_2, a_3, by the formulae

$$a_0 = \frac{2}{N}\Sigma y, \quad a_1 = \frac{2}{N}\Sigma\, y\cos\theta, \quad a_2 = \frac{2}{N}\Sigma\, y\cos 2\theta$$

$$a_3 = \frac{2}{N}\Sigma y\cos 3\theta. \quad \text{Here } N = 6 \quad \therefore \frac{2}{N} = \frac{1}{3}$$

The relevant table is as follows :

x	θ	y	$\cos\theta$	$\cos 2\theta$	$\cos 3\theta$	$y\cos\theta$	$y\cos 2\theta$	$y\cos 3\theta$
0	0	4	1	1	1	4	4	4
1	30^0	8	0.866	0.5	0	6.928	4	0
2	60^0	15	0.5	– 0.5	– 1	7.5	– 7.5	– 15
3	90^0	7	0	– 1	0	0	– 7	0
4	120^0	6	– 0.5	– 0.5	1	– 3	– 3	6
5	150^0	2	– 0.866	0.5	0	– 1.732	1	0
Σ		**42**				**13.696**	**– 8.5**	**– 5**

From the table $\Sigma y = 42$, $\Sigma y\cos\theta = 13.696$, $\Sigma y\cos 2\theta = -8.5$, $\Sigma y\cos 3\theta = -5$

$$\therefore \quad a_0 = \frac{1}{3}(42) = 14 \quad \therefore \quad \frac{a_0}{2} = 7; \; a_1 = \frac{1}{3}(13.696) \approx 4.565$$

$$a_2 = \frac{1}{3}(-8.5) \approx -2.833; \; a_3 = \frac{1}{3}(-5) \approx -1.667$$

The required values $a_0/2$, a_1, a_2, a_3 are respectively **7, 4.565, – 2.833 and – 1.667**

☞ **53.** *Obtain the constant term and the coefficients of* $\sin\theta$ *and* $\sin 2\theta$ *in the Fourier expansion of* y *given the data as below.*

θ^0	0	60	120	180	240	300	360
y	0	9.2	14.4	17.8	17.3	11.7	0

>> Here the interval of θ is $(0, 360^0)$ *i.e.*, $0 \le \theta \le 2\pi$ and the value of y at $\theta = 0$ and $\theta = 2\pi$ must be the same by the periodic property $f(\theta + 2\pi) = f(\theta)$. When the values are given both at $\theta = 0$

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and $\theta = 2\pi$ we must omit one of them. We need to compute the coefficient of $\sin\theta$ *i.e.*, b_1 and $\sin 2\theta$ *i.e.*, b_2 in the Fourier expansion of y, using the formulae

$b_1 = \frac{2}{N} \Sigma\, y \sin\theta$, $b_2 = \frac{2}{N} \Sigma\, y \sin 2\theta$, where $N = 6$ by omitting the last value. The relevant table is as follows

θ^0	y	$\sin\theta$	$\sin 2\theta$	$y\sin\theta$	$y\sin 2\theta$
0	0	0	0	0	0
60	9.2	0.866	0.866	7.9672	7.9672
120	14.4	0.866	– 0.866	12.4704	– 12.4704
180	17.8	0	0	0	0
240	17.3	– 0.866	0.866	– 14.9818	14.9818
300	11.7	– 0.866	– 0.866	– 10.1322	– 10.1322
Total				**– 4.6764**	**0.3464**

Thus we have

$$b_1 = \frac{2}{6}(-4.6764) = \mathbf{-1.5588}$$

$$b_2 = \frac{2}{6}(0.3464) = \mathbf{0.1155}$$

☞ **54.** *The following values of y and x are given. Find the Fourier series of y upto second harmonics.*

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

>> The values of y at $x = 0$ and $x = 12$ are same. Hence the interval of x is $(0, 12)$ *i.e.*, $0 \le x \le 12$ and we shall omit the value of y for $x = 12$ in the process of calculation.

The Fourier series of period $2l$ is given by

$$y = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

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Putting $l = 6$, the Fourier series up to the second harmonics is

$$y = f(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{6} + b_1 \sin\frac{\pi x}{6}\right) + \left(a_2 \cos\frac{2\pi x}{6} + b_2 \sin\frac{2\pi x}{6}\right)$$

Putting $\theta = \pi x/6$ we have

$$y = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta).$$

x	y	$\theta^0 = \pi x/6$	$\cos\theta$	$y\cos\theta$	$\cos 2\theta$	$y\cos 2\theta$
0	9.0	0	1	9	1	9.0
2	18.2	60	0.5	9.1	– 0.5	– 9.1
4	24.4	120	– 0.5	– 12.2	– 0.5	– 12.2
6	27.8	180	– 1	– 27.8	1	27.8
8	27.5	240	– 0.5	– 13.75	– 0.5	– 13.75
10	22.0	300	0.5	11.0	– 0.5	– 11.0
Total	**128.9**			**– 24.65**		**– 9.25**

$\sin\theta$	$y\sin\theta$	$\sin 2\theta$	$y\sin 2\theta$
0	0	0	0
0.866	15.7612	0.866	15.7612
0.866	21.1304	– 0.866	– 21.1304
0	0	0	0
– 0. 866	– 23.815	0.866	23.815
– 0.866	– 19.052	– 0.866	– 19.052
	– 5.9754		**– 0.6062**

The required Fourier series up to the second harmonics is given by

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$$a_0 = \frac{2}{N}\Sigma y \quad = \frac{2}{6}(128.9) \approx 42.967 \quad \therefore \frac{a_0}{2} \approx 21.4835$$

$$a_1 = \frac{2}{N}\Sigma y \cos\theta \quad = \frac{2}{6}(-24.65) \approx -8.217$$

$$a_2 = \frac{2}{N}\Sigma y \cos 2\theta \quad = \frac{2}{6}(-9.25) \approx -3.083$$

$$b_1 = \frac{2}{N}\Sigma y \sin\theta \quad = \frac{2}{6}(-5.9754) \approx -1.9918$$

$$b_2 = \frac{2}{N}\Sigma y \sin 2\theta \quad = \frac{2}{6}(-0.6062) \approx -0.202$$

$$y = f(x) = 21.4835 + \left(-8.217 \cos\frac{\pi x}{6} - 1.9918 \sin\frac{\pi x}{6}\right)$$

$$+ \left(-3.083 \cos\frac{\pi x}{3} - 0.202 \sin\frac{\pi x}{3}\right)$$

55. *The following data gives the variations of a periodic current over a period.*

t *secs*	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
A *amps*	9.0	18.2	24.4	27.8	27.5	22.0	9.0

Find numerically the direct current part of the variable current and obtain the amplitudes to the second harmonic.

>> We observe that the values of A at $t = 0$ and $t = T$ are the same. Hence we shall omit the last value. We convert $A = f(t)$ to the period 2π by putting $\theta = 2\pi(t/T)$ so that we have $\theta = 0$ when $t = 0$ and $\theta = 2\pi$ when $t = T$. The corresponding values of θ are respectively $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ, 360^\circ$,

The Fourier series upto the second harmonics is represented by

$$A = a_0/2 + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta)$$

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We prepare the relevant table considering the values of A and θ in $0 \le \theta < 2\pi$.

Since the data is same as in Example - 54 the Fourier series is given by

$$A = f(\theta) = 21.4835 + (-8.217 \cos\theta - 1.9918 \sin\theta)$$

$$+ (-3.083 \cos 2\theta - 0.202 \sin 2\theta)$$

The *direct current part* of the variable current is the constant term in the Fourier series being **21.4835**

Amplitude of the first harmonic $= \sqrt{a_1^2 + b_1^2} = \mathbf{8.455}$

Amplitude of the second harmonic $= \sqrt{a_2^2 + b_2^2} = \mathbf{3.09}$

EXERCISES

1. Analyse the following data harmonically upto the second harmonic.

x^0	0	30	60	90	120	150	180
y	134	444	326	106	94	144	66
x^0	210	240	270	300	330		
y	– 44	– 126	– 306	– 494	– 344		

2. Determine the first two harmonics in the Fourier series of y given the data

x^0	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	0.8	0.6	0.4	0.7	0.9	1.1	0.8

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