

NAME: _____ STD.: _____ SEC.: _____ ROLL NO.: _____ SUB.: APPLIED MATHEMATICS

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2.		Numerical Methods	internal choice	
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Pattern:		Part-A → 5M (no choice)		
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		Unit-1 & 2 for CIE-I		
		Access Code - FNM7 - J4M8 - JGGM8		

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UNIT-1 MATRICES

Elementary Row operations/transformations:

The following are the row operations defined on a matrix A -

- (1) $R_i \leftrightarrow R_j$ interchanging any rows
- (2) $R_i \rightarrow k R_i$, $k \neq 0$ multiplying a row with a non-zero scalar k
- (3) $R_i \rightarrow R_i + k R_j$ multiple times of a row is added to the same row.

NOTE: Two matrices are said to be equivalent matrices if one is obtained from the other using row transformations finite number of times; denoted as $A \sim B$.

Echelon form of a matrix A:

The first non-zero element of each row is called Leading Element of that row.

~~The~~ A non-zero matrix is said to be in Echelon form if it satisfies the following conditions:

- (1) Zero rows are always placed below all the non-zero rows.
- (2) All the elements below the leading element are zero.

Eg: An identity matrix, any diagonal matrix are in Echelon form.

$$A = \begin{bmatrix} 4 & 0 & 3 & 5 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Convert $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ into Echelon form.

Sol: $R_2 \rightarrow R_2 - R_1$
 $\sim$
 $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} //$$

Rank of a matrix:

Given a matrix A , reduce the matrix A to its Echelon form. Then, the number of non-zero rows in Echelon form is called as Rank of the matrix A , denoted as $\rho(A)$.

2) Determine rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Sol: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ \sim \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 5R_2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}$$

$\therefore$ Rank of A is 3 //

3) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ \sim \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{5}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & 0 & 12 \end{bmatrix}$$

$\therefore$ Rank is 3 //

$$7 - \frac{5}{2}(-2) \quad 7 + 5 =$$

4) $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -3/2 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 + \frac{1}{2} R_1$$

$$\sim \begin{bmatrix} 4 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank is 1 //

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5) Find the value of 'b' such that rank of matrix $A = 3$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$$

Sol: $R_2 \rightarrow R_2 - 4R_1$ $R_3 \rightarrow R_3 - bR_1$ $R_4 \rightarrow R_4 - 9R_1$ $\sim$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & b+9 & 3 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\sim$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b+9 & 3 \end{bmatrix}$

$R_4 \rightarrow R_4 - (b+9)R_3$ $\sim$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -b-6 \end{bmatrix}$

Given $\rho(A) = 3$

$\Rightarrow -b-6 = 0$

$\Rightarrow \boxed{b = -6}$

New, if $b = 2$, $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

$R_3 \rightarrow R_3 - \frac{1}{4}R_2$ $\sim$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & -8 \end{bmatrix}$

$$R_4 \rightarrow R_4 + 16R_3 \quad \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3 \quad \therefore \boxed{b=2}$$

$$\text{If } b = -2 \text{ then } A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \begin{array}{l} \rho(A) = 4 \\ \therefore b \neq -2 \end{array}$$

System of Equations:

Consider a system of equations as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$\vdots$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

where $x_1, x_2, \dots, x_n$ are set of unknowns.

In matrix form, the above system of equations can be written as

$$AX = B \quad \text{where } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = \text{coefficient matrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

NOTE: ① If RHS, B is zero, then the system of equations are called Homogeneous system of equation. If $B \neq 0$, then it is known as non-homogeneous system of equations.

② A system of equations is said to be consistent if, it has a solution.

③ Homogeneous system of equations is always consistent (all $x_i = 0$).

condition for the consistency -

- (1) Given $XA = B$.
- (2) We form an augmented matrix $[A : B]$ or $[A | B]$
- (3) compute (Find) the rank of augmented matrix $[A : B]$, find the rank of $[A]$
- (4) The system is consistent iff. $\rho[A : B] = \rho(A)$
- (5) Let $\rho(A) = \rho(A : B) = r$ and n be the number of unknown then, if $r = n$, the system has unique solution.
if $r < n$, the system has infinitely many solutions.
(where ' $n-r$ ' variables are taken arbitrarily)

1) Test the consistency of the given system of equations using Gauss Elimination method. If consistent, find the solution.

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ 2x_1 + 5x_2 - 2x_3 &= 3 \\ x_1 + 7x_2 - 7x_3 &= 5 \end{aligned}$$

Sol:

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{array} \right]$$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - R_1 \end{aligned} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{array} \right]$$

$$\rho(A : B) = 3 \neq \rho(A) = 2$$

$\therefore$ The system is inconsistent.

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(2) $x + y + z = 4$

$2x + y - z = 1$

$x - y + 2z = 2$

Sol: $[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 2 & 1 & -1 & : & 1 \\ 1 & -1 & 2 & : & 2 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$
 $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & -1 & -3 & : & -7 \\ 0 & -2 & 1 & : & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & -1 & -3 & : & -7 \\ 0 & 0 & 7 & : & 12 \end{bmatrix}$$

$\rho(A : B) = 3 = \rho(A) = \text{no. of unknowns}$
Hence ^{unique} solution exists

In equation form, $A'X = B'$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 12 \end{bmatrix}$$

The equivalent system is

$x + y + z = 4$

$-y - 3z = -7$

$7z = 12$

By back substitution, $z = \frac{12}{7}$

$-y = -7 + \frac{36}{7} = \frac{13}{7}$

$x = 4 - \frac{13}{7} - \frac{12}{7} = \frac{28 - 13 - 12}{7} = \frac{3}{7}$

∴ The solution is

$$\begin{cases} x = 5/7 \\ y = 13/7 \\ z = 12/7 \end{cases}$$

3) $2x_1 + x_2 + 4x_3 = 12$

$4x_1 + 11x_2 - x_3 = 33$

$8x_1 - 3x_2 + 2x_3 = 20$

Sol:

$$[A : B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array} \quad N \quad \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix}$$

$\frac{33}{24} \div 9$
 $-14 \div 7 = -2$

$$R_3 \rightarrow R_3 + 7R_2 \quad N \quad \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & 0 & -21 & : & -21 \end{bmatrix}$$

$-28 \div 7 = -4$

$\rho(A : B) = 3 = \rho(A)$

∴ Solution exists and is unique.

In equation form: $A'X = B'$

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & 9 & -9 \\ 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ -21 \end{bmatrix}$$

Equivalent form,

$2x_1 + x_2 + 4x_3 = 12$

$9x_2 = 9 + 9$

$9x_2 - 9x_3 = 9$

$x_2 = \frac{18}{9} = 2$

$-21x_3 = -21$

$\Rightarrow \boxed{x_3 = 1}$

$2x_1 = 12 - 4 - 2$

$x_1 = \frac{6}{2} = 3$

$$\therefore \text{Solution is } \begin{bmatrix} x_1 = 3 \\ x_2 = 2 \\ x_3 = 1 \end{bmatrix}$$

4) $5x + 3y + 7z = 4$
 $3x + 26y + 2z = 9$
 $7x + 2y + 10z = 5$

Sol: $[A : B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$

$$R_2 \rightarrow R_2 - \frac{3}{5}(R_1) \quad N \quad \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 12\frac{1}{5} & -11\frac{1}{5} & : & 33\frac{3}{5} \\ 0 & -11\frac{1}{5} & 1\frac{1}{5} & : & -3\frac{3}{5} \end{bmatrix}$$

$$R_3 \rightarrow R_3 + \frac{1}{11}(R_2) \quad \sim \quad \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 12\frac{1}{5} & -11\frac{1}{5} & : & 33\frac{3}{5} \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A : B) = 2 = \rho(A) = n$$

But number of unknowns $= 3 = n$
 $\Rightarrow n < n$

Hence infinitely many solutions exist.

$\therefore$ The equivalent system is

$$5x + 3y + 7z = 4$$

$$\frac{121y}{5} - \frac{11z}{5} = \frac{33}{5}$$

Taking $n - n = 3 - 2 = 1$ variable as arbitrary,

Let $z = k$ be arbitrary, ~~the equivalent~~

$$\frac{121y}{5} = \frac{33}{5} + \frac{11k}{5} \quad \text{or} \quad y = \frac{33 + 11k}{121}$$

$$5x + 3y = 4 - 7k \quad \Rightarrow \quad 5x = 4 - 7k - \frac{99}{121} - \frac{33k}{121}$$

$$5x = \frac{4-99}{121} - \left(\frac{7+33}{121} \right) k$$

$$x = \frac{35-8k}{55} = \frac{7-16k}{11}$$

$$\therefore \text{The solution is } \begin{cases} x = \frac{7-16k}{11} \\ y = \frac{3+k}{11} \\ z = k \end{cases} //$$

5) $x - y + z = 2$

$$3x - y + 2z = -6$$

$$3x + y + z = -18$$

Sol:

$$[A : B] = \begin{bmatrix} 1 & -1 & 1 & : & 2 \\ 3 & -1 & 2 & : & -6 \\ 3 & 1 & 1 & : & -18 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \sim \begin{bmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 2 & -1 & : & -12 \\ 0 & 4 & -2 & : & -24 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2 \sim \begin{bmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 2 & -1 & : & -12 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A : B) = 2 = \rho(A) = n; \quad n = 3; \quad n < n$$

$\therefore$ Infinitely many solutions exist.

The equivalent system is

$$x - y + z = 2$$

$$2y - z = -12$$

Let ~~z~~ $z = k$ ($n - n$) = $(3 - 2) = 1$ variable be arbitrary

$$\text{Let } z = k$$

$$2y = -12 + k$$

$$y = \frac{-6+k}{2}$$

$$x = 2 + \left(-6 + \frac{k}{2}\right) - k = 2 - 6 + \frac{k}{2} - k = -4 - \frac{k}{2}$$

$$x = \frac{-8 - k}{2}$$

$$\therefore \text{Solution is } \begin{bmatrix} x = -4 - \frac{1}{2}k \\ y = -6 + \frac{k}{2} \\ z = k \end{bmatrix} //$$

6) For what values of constant 'k', the given system of equations is consistent. Also find the solution in each case.

$$x + y + z = 1$$

$$x + 2y + 4z = K$$

$$x + 4y + 10z = K^2$$

Sol: $[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & K \\ 1 & 4 & 10 & : & K^2 \end{bmatrix}$

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & K-1 \\ 0 & 3 & 9 & : & K^2-1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2 \sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & K-1 \\ 0 & 0 & 0 & : & K^2-3K+2 \end{bmatrix}$$

If $K^2 - 3K + 2 \neq 0$, then $\rho(A:B) = 3$, $\rho(A) = 2$
Hence inconsistent

The system is consistent only when $K^2 - 3K + 2 = 0$ such that
 $\rho(A:B) = \rho(A) = 2$

$\therefore K=2$ and $K=1$

Case: ①, when $K=1$

The equivalent system is

$$x + y + z = 1$$

$$y + 3z = 0$$

Take $3-2=1$, variable arbitrary.

Let $z = p$

$$\Rightarrow y = -3p$$

$$x = 1 - p - (-3p)$$

$$x = 1 + 2p$$

$$\therefore \text{Solution is } \begin{bmatrix} x = 1 + 2p \\ y = -3p \\ z = p \end{bmatrix}$$

Case: ②, when $K=2$

The equivalent system is

$$x + y + z = 1$$

$$y + 3z = 1$$

Take $3-2=1$ variable as arbitrary

Let $z = q$

$$\Rightarrow y = 1 - 3q$$

$$x = 1 - 1 + 3q - q = 2q$$

$$\therefore \text{Solution is } \begin{bmatrix} x = 2q \\ y = 1 - 3q \\ z = q \end{bmatrix} //$$

7) Prove that the given system of equations do not have a solution unless $a+c=2b$.

$$3x + 4y + 5z = a$$

$$4x + 5y + 6z = b$$

$$5x + 6y + 7z = c$$

Sol: $[A:B] = \begin{bmatrix} 3 & 4 & 5 & : & a \\ 4 & 5 & 6 & : & b \\ 5 & 6 & 7 & : & c \end{bmatrix}$

$$5 - \frac{20}{3} = \frac{15-20}{3}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - \frac{4}{3} R_1 \\ R_3 \rightarrow R_3 - \frac{5}{3} R_1 \end{array} \sim \begin{bmatrix} 3 & 4 & 5 & : & a \\ 0 & -\frac{1}{3} & -\frac{2}{3} & : & b - \frac{4}{3}a \\ 0 & -\frac{2}{3} & -\frac{4}{3} & : & c - \frac{5}{3}a \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2 \sim \begin{bmatrix} 3 & 4 & 5 & : & a \\ 0 & -\frac{1}{3} & -\frac{2}{3} & : & b - \frac{4}{3}a \\ 0 & 0 & 0 & : & a - 2b + c \end{bmatrix}$$

$$\rho(A:B) = 3 \quad ; \quad \rho(A) = 2$$

The system is consistent only when $a - 2b + c = 0 \Rightarrow a + c = 2b //$
Hence proved//

8) Investigate the values of λ and μ such that the given set of equations
(i) has no solution (ii) has unique solution (iii) has infinitely many solutions.

(a) $2x + 3y + 5z = 9$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

Sol: $[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix}$

$$R_2 \rightarrow R_2 - \frac{7}{2} R_1$$

$$\sim \begin{bmatrix} 2 & 2 & 5 & : & 9 \\ 0 & -15/2 & -39/2 & : & -49/2 \\ 0 & 0 & \lambda-5 & : & \mu-9 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_1$$

$$\lambda=5 \Rightarrow \rho(A)=2 \quad \mu \neq 9 \Rightarrow \rho(B)=3$$

- (i) For $\lambda=5$ and $\mu \neq 9$, system has no solution.
 (ii) For $\lambda \neq 5$ and for any value of μ , system has unique solution.
 (iii) For $\lambda=5$ and $\mu=9$, system has infinitely many solutions.
 i.e. $n=2, m=3; 2 < 3$

b) $x + y + z = 6$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

Sol:-

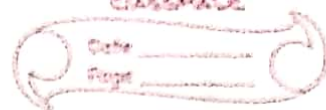
$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda-1 & : & \mu-6 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2 \sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda-3 & : & \mu-10 \end{bmatrix}$$

- (i) For $\lambda=3$ and $\mu \neq 10$, system has no solution.
 (ii) For $\lambda \neq 3$ and for any value of μ , system has unique solution.
 (iii) For $\lambda=3$ and $\mu=10$, system has infinitely many solutions.

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- 9) Determine the values of a & b for which the given system has
 (i) no solution (ii) unique solution (iii) infinitely many solutions

Sol:

$$\begin{aligned} x + 2y + 3z &= 6 \\ x + 3y + 5z &= 9 \\ 2x + 5y + az &= b \end{aligned}$$

Sol:

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 3 & 5 & : & 9 \\ 2 & 5 & a & : & b \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 - R_1 & R_3 &\rightarrow R_3 - 2R_1 \\ & \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 1 & a-6 & : & b-12 \end{bmatrix} \end{aligned}$$

$$R_3 \rightarrow R_3 - R_2 \sim \begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & a-8 & : & b-15 \end{bmatrix}$$

- (i) For $a=8$ and $b \neq 15$, system has no solution
 (ii) For $a \neq 8$ and for any value of b , system has unique solution
 (iii) For $a=8$ and $b=15$, system has infinitely many solutions.

- 10) For what value of λ , the given system is consistent. Find the solution in each case.

$$3x - y + 4z = 3$$

$$x + 2y - 3z = -2$$

$$6x + 5y + \lambda z = -3$$

Sol:

$$[A:B] = \begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 1 & 2 & -3 & : & -2 \\ 6 & 5 & \lambda & : & -3 \end{bmatrix}$$

$$\begin{aligned} R_2 &\rightarrow R_2 - \frac{1}{3}R_1 & R_3 &\rightarrow R_3 - 2R_1 \\ & \sim \begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 0 & 7/3 & -13/3 & : & -3 \\ 0 & 7 & \lambda-8 & : & -9 \end{bmatrix} \end{aligned}$$

$$R_3 \rightarrow R_3 - 3R_2 \sim \begin{bmatrix} 3 & -1 & 4 & : & 3 \\ 0 & +7/3 & -13/3 & : & -3 \\ 0 & 0 & \lambda+5 & : & 0 \end{bmatrix}$$

$$\Rightarrow \rho(A:B) = \rho(A) = 3 = 3 \text{ if } \lambda \neq -5;$$

$\therefore$ unique solution exists

The equivalent system,

$$3x - y + 4z = 3$$

$$\frac{7}{3}y - \frac{13}{3}z = -3$$

$$(\lambda+5)z = 0$$

$$\Rightarrow z = 0$$

$$\Rightarrow \frac{7}{3}y = -3 \Rightarrow y = \frac{-9}{7}$$

$$\Rightarrow 3x - \left(\frac{-9}{7}\right) = 3 \Rightarrow 3x = 3 - \frac{9}{7} = \frac{21-9}{7} = \frac{12}{7}$$

$$\Rightarrow x = \frac{30}{7} \cdot \frac{1}{3} = \frac{10}{7} \quad x = \frac{4}{7}$$

$\therefore$ solution is

$$\begin{cases} x = 4/7 \\ y = -9/7 \\ z = 0 \end{cases}$$

$$\Rightarrow \rho(A:B) = \rho(A) = 2 < 3 \text{ if } \lambda = -5;$$

$\therefore$ Infinitely many solutions exist.

Gauss-Seidel Method

This method is used to find the solution of system of equations numerically.

This method converges to the solution rapidly if the system is diagonally dominant.

We start with an initial approximation $x^{(0)}, y^{(0)}, z^{(0)}$
[$x^{(0)}=0, y^{(0)}=0, z^{(0)}=0$ unless specified] and proceed step-by-step towards a new solution by using the equations from the diagonal dominant system (performing iterations of x, y, z).

The iterations are stopped when the two successive iterated values of x, y, z are same upto desired accuracy.

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1) Solve the given system of equations using Gauss-Seidel method.

$$x + 3y + 10z = 24$$

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

Sol. - Rearranging it in diagonal dominant form -

$$28x + 4y - z = 32 \quad \text{--- (1)}$$

$$2x + 17y + 4z = 35 \quad \text{--- (2)}$$

$$x + 3y + 10z = 24 \quad \text{--- (3)}$$

From eq. (1), $32 - 4y + z = 28x$

$$x = \frac{1}{28} [32 - 4y + z] \quad \text{--- (4)}$$

From eq. (2), $y = \frac{1}{17} [35 - 2x - 4z] \quad \text{--- (5)}$

From eq. (3), $z = \frac{1}{10} [24 - 3y - x] \quad \text{--- (6)}$

Let the initial approximation be $x^{(0)}=0, y^{(0)}=0, z^{(0)}=0$
From eq. (4), $x^{(1)} = \frac{1}{28} [32 - 4y^{(0)} + z^{(0)}]$

$$\approx 1.1428$$

$$\text{from eq. (5), } y^{(1)} = \frac{1}{17} [35 - 2x^{(1)} - 4z^{(0)}]$$

$$= \frac{1}{17} [35 - 2(1.1428)]$$

$$\approx 1.9243 //$$

$$\text{from eq. (6), } z^{(1)} = \frac{1}{10} [24 - x^{(1)} - 3y^{(1)}]$$

$$= \frac{1}{10} [24 - 1.1428 - 3(1.9243)]$$

$$\approx 1.7084 //$$

$$x^{(2)} = \frac{1}{28} [32 - 4y^{(1)} + z^{(1)}]$$

$$= \frac{1}{28} [32 - 4(1.9243) + 1.7084]$$

$$x^{(2)} \approx 0.9289 //$$

$$y^{(2)} = \frac{1}{17} [35 - 2x^{(2)} - 4z^{(1)}] = \frac{1}{17} [35 - 2(\overset{0.9289}{\cancel{1.9243}}) - 4(1.7084)]$$

$$y^{(2)} \approx 1.5475 //$$

$$z^{(2)} = \frac{1}{10} [24 - x^{(2)} - 3y^{(2)}] = \frac{1}{10} [24 - 0.9289 - 3(1.5475)]$$

$$z^{(2)} \approx 1.8428 //$$

$$x^{(3)} = \frac{1}{28} [32 - 4y^{(2)} + z^{(2)}] = \frac{1}{28} [32 - 4(1.5475) + 1.8428]$$

$$x^{(3)} \approx 0.9876 //$$

$$y^{(3)} = \frac{1}{17} [35 - 2x^{(3)} - 4z^{(2)}] = \frac{1}{17} [35 - 2(0.9876) - 4(1.8428)]$$

$$y^{(3)} \approx 1.5090 //$$

$$z^{(2)} = \frac{1}{10} [24 - x^{(3)} - 3y^{(3)}] = \frac{1}{10} [24 - 0.9876 - 3(1.5090)]$$

$$z^{(2)} \approx 1.8485 //$$

$$2) \quad 3x + 20y - z = -18$$

$$20x + y - 2z = 17$$

$$2x - 3y + 20z = 25$$

Sol: Rearranging the equations -

$$20x + y - 2z = 17 \quad \text{--- (1)}$$

$$3x + 20y - z = -18 \quad \text{--- (2)}$$

$$2x - 3y + 20z = 25 \quad \text{--- (3)}$$

$$\text{from eq. (1), } x = \frac{1}{20} [17 - y + 2z] \quad \text{--- (4)}$$

$$\text{from eq. (2), } y = \frac{1}{20} [-18 - 3x + z] \quad \text{--- (5)}$$

$$\text{eq. (3), } z = \frac{1}{20} [25 - 2x + 3y] \quad \text{--- (6)}$$

Taking the initial approximations as $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$

$$x^{(1)} = \frac{1}{20} [17 - 0 + 2(0)] \approx 0.8500 //$$

$$y^{(1)} = \frac{1}{20} [-18 - 3x^{(1)} + z^{(0)}] = \frac{1}{20} [-18 - 3(0.85) + 0]$$

$$y^{(1)} \approx -1.0275 //$$

$$z^{(1)} = \frac{1}{20} [25 - 2(0.85) + 3(-1.0275)]$$

$$z^{(1)} \approx 1.0108 //$$

$$x^{(2)} = \frac{1}{20} [17 + 1.0275 + 2(1.0108)]$$

$$x^{(2)} \approx 1.0024 //$$

$$y^{(2)} = \frac{1}{20} [-18 - 3(1.0024) + 1.0108]$$

$$y^{(2)} \approx -0.9998 //$$

$$z^{(2)} = \frac{1}{20} [25 - 2(1.0024) + 3(-0.9998)]$$

$$z^{(2)} \approx 0.9997 //$$

$$x^{(3)} = \frac{1}{20} [17 - (-0.9998) + 2(0.9997)]$$

$$x^{(3)} \approx 0.9999 //$$

$$y^{(3)} = \frac{1}{20} [-18 - 3(0.9999) + 0.9997]$$

$$y^{(3)} = -1.0000 //$$

$$z^{(3)} = \frac{1}{20} [25 - 2(0.9999) + 3(-1.0000)]$$

$$z^{(3)} = 1.0000 //$$

$$x^{(4)} = \frac{1}{20} [17 - (-1) + 2(1)]$$

$$x^{(4)} = 1.0000 //$$

$$y^{(4)} = \frac{1}{20} [-18 - 3(-1) + (-1)]$$

$$y^{(4)} = -1.0000 //$$

$$z^{(4)} = \frac{1}{20} [25 - 2(1) + 3(-1)]$$

$$z^{(4)} = 1.0000 //$$

The solution is $x=1, y=-1, z=1$ //

$$(3) \quad 6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

$$27x + 6y - z = 85$$

Sol: Rearranging $\Rightarrow$

$$\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned}$$

$$\Rightarrow x = \frac{1}{27} [85 - 6y + z]$$

$$y = \frac{1}{15} [72 - 6x - \frac{2z}{54}]$$

$$z = \frac{1}{54} [110 - x - y]$$

Initial approximations $x^{(0)}=0, y^{(0)}=0, z^{(0)}=0$

$$x^{(1)} = \frac{1}{27} [85 - 0 + 0] \approx 3.1481 //$$

$$y^{(1)} = \frac{1}{15} [72 - 6(3.1481) - 2(0)] \approx 3.5407 //$$

$$z^{(1)} = \frac{1}{54} [110 - 3.1481 - 3.5407] \approx 1.9131 //$$

$$x^{(2)} = \frac{1}{27} [85 - 6(3.5407) + 1.9131] \approx 2.4321 //$$

$$y^{(2)} = \frac{1}{15} [72 - 6(2.4321) - 2(1.9131)] \approx 3.5720 //$$

$$z^{(2)} = \frac{1}{54} [110 - 2.4321 - 3.5720] \approx 1.9258 //$$

$$x^{(3)} = \frac{1}{27} [85 - 6(3.5720) + \overset{2.4256}{\cancel{2.2117}} (1.9258)] \approx \cancel{2.2117} //$$

$$y^{(3)} = \frac{1}{15} [72 - 6(2.2117) - 2(1.9258)] \approx 3.6585 //$$

$$z^{(3)} = \frac{1}{54} [110 - 2.4256 - 3.6585] \approx 1.9243 //$$

$$4) \begin{aligned} 4x + 2y + z &= 11 \\ -x + 2y &= 3 \\ 2x + y + 4z &= 16 \end{aligned}$$

$$5) \begin{aligned} 10x + y + z &= 12 \\ 2x + 2y + 10z &= 14 \\ 2x + 10y + z &= 13 \end{aligned}$$

$$6) \begin{aligned} 7x + 52y + 13z &= 104 \\ 3x + 8y + 29z &= 71 \\ 83x + 11y - 4z &= 75 \end{aligned}$$

$$7) \begin{aligned} 5x - y + z &= 10 \\ 2x + 4y &= 12 \\ x + y + 5z &= -1 \end{aligned}$$

Start with $(2, 2, -1)$

$$4) \begin{aligned} 4x + 2y + z &= 11 \quad \text{--- (1)} \\ -x + 2y + 0 &= 3 \quad \text{--- (2)} \\ 2x + y + 4z &= 16 \quad \text{--- (3)} \end{aligned}$$

$$\text{from (1)} \Rightarrow x = \frac{1}{4} [11 - 2y - z] \quad \text{--- (4)}$$

$$\text{from (2)} \Rightarrow y = \frac{1}{2} [3 + x] \quad \text{--- (5)}$$

$$\text{from (3)} \Rightarrow z = \frac{1}{4} [16 - 2x - y] \quad \text{--- (6)}$$

Let the initial approximations be $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$

$$x^{(1)} = \frac{1}{4} [11 - 2y^{(0)} - z^{(0)}] \approx 2.7500$$

$$y^{(1)} = \frac{1}{2} [3 + x^{(1)}] = \frac{1}{2} [3 + 2.75] \approx 2.8750$$

$$z^{(1)} = \frac{1}{4} [16 - 2x^{(1)} - y^{(1)}] = \frac{1}{4} [16 - 2(2.75) - 2.875] \approx 1.90625$$

$$x^{(2)} = \frac{1}{4} [11 - 2y^{(1)} - z^{(1)}] \approx 0.8359$$

$$y^{(2)} = \frac{1}{2} [3 + x^{(2)}] \approx 1.9179$$

$$z^{(2)} = \frac{1}{4} [16 - 2x^{(2)} - y^{(2)}] = 2.1025$$

$$x^{(3)} = 1.0154 \quad \text{EPPF.3} = [x^{(4)} = 0.9985]$$

$$y^{(3)} = 2.0077$$

$$y^{(4)} = 1.9992$$

$$z^{(3)} = 2.99031000.1 = [z^{(4)} = 3.0009]$$

$$x^{(5)} = 0.9985$$

$$y^{(5)} = 1.999200.1 = [y^{(5)} = 1.9992]$$

$$z^{(5)} = 3.0009$$

$$x^{(5)} = 0.9995$$

$$y^{(5)} = 1.9992$$

$$z^{(5)} = 3.0009$$

5) Rearranging in diagonally dominant form -

$$10x + y + z = 12 \quad \text{--- (1)}$$

$$2x + 10y + z = 13 \quad \text{--- (2)}$$

$$2x + 2y + 10z = 14 \quad \text{--- (3)}$$

$$\text{from eq. (1)} \Rightarrow x = \frac{1}{10} [12 - y - z] \quad \text{--- (4)}$$

$$\text{from eq. (2)} \Rightarrow y = \frac{1}{10} [13 - 2x - z] \quad \text{--- (5)}$$

$$\text{from eq. (3)} \Rightarrow z = \frac{1}{10} [14 - 2x - 2y] \quad \text{--- (6)}$$

Taking initial approximations as $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$

$$\text{in eq. (4), } x^{(1)} = \frac{1}{10} [12 - y^{(0)} - z^{(0)}] = \frac{12}{10} = 1.2000 //$$

$$y^{(1)} = \frac{1}{10} [13 - 2x^{(1)} - z^{(0)}] = \frac{1}{10} [13 - 2(1.2)] = 1.0600 //$$

$$z^{(1)} = \frac{1}{10} [14 - 2x^{(1)} - 2y^{(1)}] = \frac{1}{10} [14 - 2(1.2) - 2(1.06)] = 0.9480 //$$

$$x^{(2)} = \frac{1}{10} [12 - y^{(1)} - z^{(1)}] = 0.9992 //$$

$$y^{(2)} = \frac{1}{10} [13 - 2x^{(2)} - z^{(1)}] = 1.0053 //$$

$$z^{(2)} = \frac{1}{10} [14 - 2x^{(2)} - 2y^{(2)}] = 0.9991 //$$

$$x^{(3)} = \frac{1}{10} [12 - y^{(2)} - z^{(2)}] = 0.9995 //$$

$$y^{(3)} = \frac{1}{10} [13 - 2x^{(3)} - z^{(2)}] = 1.0001 //$$

$$z^{(3)} = \frac{1}{10} [14 - 2x^{(3)} - 2y^{(3)}] = 1.0000 //$$

$$x^{(4)} = \frac{1}{10} [12 - y^{(3)} - z^{(3)}] = 0.9999 //$$

$$y^{(4)} = \frac{1}{10} [13 - 2x^{(4)} - z^{(3)}] = 1.0000 //$$

$$z^{(4)} = \frac{1}{10} [14 - 2x^{(4)} - 2y^{(4)}] = 1.0000 //$$

$\therefore$ The solution is $x = 0.9$, $y = 1$, $z = 1$ //

e) Rearranging in diagonally dominant form -

$$83x + 11y - 4z = 95 \quad \text{--- (1)}$$

$$7x + 52y + 13z = 104 \quad \text{--- (2)}$$

$$3x + 8y + 29z = 71 \quad \text{--- (3)}$$

$$\text{from eq. (1)} \Rightarrow x = \frac{1}{83} [95 - 11y + 4z] \quad \text{--- (4)}$$

$$\text{from eq. (2)} \Rightarrow y = \frac{1}{52} [104 - 7x - 13z] \quad \text{--- (5)}$$

$$\text{from eq. (3)} \Rightarrow z = \frac{1}{29} [71 - 3x - 8y] \quad \text{--- (6)}$$

Taking initial approximations as $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$

$$x^{(1)} = \frac{1}{83} [95 - 11y^{(0)} - 4z^{(0)}] = \frac{95}{83} \approx 1.1445 //$$

$$y^{(1)} = \frac{1}{52} [104 - 7x^{(1)} - 13z^{(0)}] \approx 1.8459 //$$

$$z^{(1)} = \frac{1}{29} [71 - 3x^{(1)} - 8y^{(1)}] \approx 1.8206 //$$

$$x^{(2)} = \frac{1}{83} [95 - 11y^{(1)} - 4z^{(1)}] = 0.8122 //$$

$$y^{(2)} = \frac{1}{52} [104 - 7x^{(2)} - 13z^{(1)}] = 1.4355 //$$

$$z^{(2)} = \frac{1}{29} [71 - 3x^{(2)} - 8y^{(2)}] = 1.9682 //$$

$$x^{(3)} = \frac{1}{83} [95 - 11y^{(2)} - 4z^{(2)}] = 0.8594 //$$

$$y^{(3)} = \frac{1}{52} [104 - 7x^{(3)} - 13z^{(2)}] = 1.3922 //$$

$$z^{(3)} = \frac{1}{29} [71 - 3x^{(3)} - 8y^{(3)}] = 1.9753 //$$

$$x^{(4)} = \frac{1}{83} [95 - 11y^{(3)} - 4z^{(3)}] = 0.8648 //$$

$$y^{(4)} = \frac{1}{52} [104 - 7x^{(4)} - 13z^{(3)}] = 1.3897 //$$

$$z^{(4)} = \frac{1}{29} [71 - 3x^{(4)} - 8y^{(4)}] = 1.9754 //$$

∴ The solution is $x = 0.8$, $y = 1.3$, $z = 1.97$

$$7) \quad 5x - y + z = 10 \quad \text{--- (1)}$$

$$8x + 4y = 12 \quad \text{--- (2)}$$

$$x + y + 5z = -1 \quad \text{--- (3)}$$

$$\Rightarrow x = \frac{1}{5} [10 + y - z] \quad \text{--- (4)}$$

$$\Rightarrow y = \frac{1}{4} [12 - 2x] \quad \text{--- (5)}$$

$$\Rightarrow z = \frac{1}{5} [-1 - x - y] \quad \text{--- (6)}$$

Taking initial approximations as $x^{(0)} = 2$, $y^{(0)} = 2$, $z^{(0)} = -1$

$$x^{(1)} = \frac{1}{5} [10 + y^{(0)} - z^{(0)}] = \frac{1}{5} [10 + 2 + 1] \approx 2.6000$$

$$y^{(1)} = \frac{1}{4} [12 - 2x^{(1)}] = \frac{1}{4} [12 - 2(2.6)] \approx 1.7000$$

$$z^{(1)} = \frac{1}{5} [-1 - x^{(1)} - y^{(1)}] = \frac{1}{5} [-1 - 2.6 - 1.7] \approx -1.0600$$

$$x^{(2)} = \frac{1}{5} [10 + y^{(1)} - z^{(1)}] = \frac{1}{5} [10 + 1.7 + 1.06] \approx 2.5520$$

$$y^{(2)} = \frac{1}{4} [12 - 2x^{(2)}] = \frac{1}{4} [12 - 2(2.552)] \approx 1.7240$$

$$z^{(2)} = \frac{1}{5} [-1 - x^{(2)} - y^{(2)}] = \frac{1}{5} [-1 - 2.5520 - 1.7240] \approx -1.0552$$

$$x^{(3)} = \frac{1}{5} [10 + y^{(2)} - z^{(2)}] = \frac{1}{5} [10 + 1.724 + 1.0552] \approx 2.5558$$

$$y^{(3)} = \frac{1}{4} [12 - 2x^{(3)}] = \frac{1}{4} [12 - 2(2.5558)] \approx 1.7221$$

$$z^{(3)} = \frac{1}{5} [-1 - x^{(3)} - y^{(3)}] = \frac{1}{5} [-1 - 2.5558 - 1.7221] \approx -1.0555$$

$$x^{(4)} = \frac{1}{5} [10 + y^{(3)} - z^{(3)}] \approx 2.5555 //$$

$$y^{(4)} = \frac{1}{4} [12 - 2x^{(4)}] \approx 1.7222 //$$

$$z^{(4)} = \frac{1}{5} [-1 - x^{(4)} - y^{(4)}] \approx -1.0555 //$$

$\therefore$ The solution is $x = 2.55$, $y = 1.72$, $z = -1.05 //$

17/8/19

Eigen values and Eigen Vectors -

Given a square matrix A of order n . If there exist, a scalar λ and a non-zero vector X such that, $AX = \lambda X$ then, the λ is called the Eigen value of matrix A and X is called the Eigen vector of matrix A corresponding to the eigen value λ .

$$AX = \lambda X \text{ is equivalently written as } [A - \lambda I]X = 0. \text{ --- (1)}$$

This represents a system of homogeneous equations and a non-trivial solution exist only when $|A - \lambda I| = 0$ --- (2)

The eq. (2) represents the characteristic equation of matrix A , and the roots of this equation are called Eigen values or characteristics values.

For each value of λ there exist an eigen vector X , $X \neq 0$.

Note: The number of eigen values is equal to the order of the matrix A . Eigen vector (X) is not unique.

1) Find the eigen values & eigen vectors of the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Sol: The characteristic equation of mat A is $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - 6\lambda - \lambda + 6 = 0$$

$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\Rightarrow \lambda_1 = 6 \quad \lambda_2 = 1 \quad \left. \vphantom{\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}} \right\} \text{ are the Eigen values of } A.$$

To find the eigen vector when $\lambda_1 = 6$

The system of equation is $[A - \lambda I]X = 0$

$$\Rightarrow [A - 6I]X = 0$$

$$\Rightarrow [A - \lambda I] = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 4x_2 = 0$$

$$x_1 - 4x_2 = 0$$

$$\Rightarrow \boxed{4x_2 = x_1}$$

Converting $[A]$ to Echelon form. $\Rightarrow \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \xrightarrow[R_2 \rightarrow R_2 + R_1]{\sim} \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rho(A) = 1, \quad n = 2$$

$2 - 1 = 1$ variable is taken as arbitrary.

$$\text{Let } x_2 = K_1$$

$$\text{then } x_1 = 4x_2 = 4K_1$$

$$\text{The eigen vector is } \begin{bmatrix} x_1 = 4K_1 \\ x_2 = K_1 \end{bmatrix} = \begin{bmatrix} 4K_1 \\ K_1 \end{bmatrix}$$

$$\therefore \text{ for } \lambda_1 = 6, \quad X_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

when $\lambda_2 = 1$

$$[A - I]X = 0 \Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 - \frac{1}{4}R_1} \sim \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$f(A) = 1, n = 2$$

$2 - 1 = 1$ variable is taken as arbitrary.

$$\text{Let } x_2 = k_1$$

$$\text{Then } 4x_1 + 4x_2 = 0$$

$$x_1 = -x_2 \Rightarrow x_1 = -k_1 \text{ and } x_2 = k_1$$

$$\therefore \text{ for } \lambda_2 = 1, X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

≡ For a 3×3 matrix, $|A - \lambda I| = 0$ can be equivalently written as

$$\lambda^3 - \left[\text{sum of principle diagonal elements of } A \right] \lambda^2 + \left[M_{11} + M_{22} + M_{33} \right] \lambda - |A| = 0$$

$$2) A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|A| = 1(5-1) - 1(1-3) + 3(1-5)$$

$$= 4 + 2 - 12 = -36$$

Sol: $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

$$|A - \lambda I| = \lambda^3 - 7\lambda^2 + [(5-1) + (1-3) + (5-1)]\lambda -$$

$$= \lambda^3 - 7\lambda^2 + (0)\lambda + 36 = 0$$

$$= \lambda^3 - 7\lambda^2 + 36 = 0$$

$$\Rightarrow \lambda_1 = -2, \lambda_2 = 3, \lambda_3 = 6$$

20/9/19

Solution for homogeneous system $AX=0$

convert A' into Echelon form

$\rho(A') = n < n$ (no. of unknowns) $\Rightarrow n-k$ variables = $n-k$ free variables

To find eigen vectors -

for $\lambda_1 = -2$

The equivalent matrix is $[A - \lambda I]X = 0$

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 - 3R_1, R_3 - 3R_1} \begin{bmatrix} 1 & 7 & 1 \\ 0 & -20/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The equivalent system of eq. is.

$$3x_1 + x_2 + 3x_3 = 0$$

$$\frac{-20}{3} x_2 = 0 \Rightarrow x_2 = 0$$

$$\Rightarrow 3x_1 + 3x_3 = 0$$

$$\Rightarrow x_1 = -x_3$$

$n-k = 3-2 = 1$ variable be arbitrary

let $x_3 = k_1$ be arbitrary

$$\therefore x_1 = -x_3 = -k_1$$

$$x_2 = 0$$

$$\therefore \text{The Eigen vector is } x_1 = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda_2 = 6$

The equivalent system is $[A - \lambda I]X = 0$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & -1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 8/5 & -16/5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4/5 & 8/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A') = 2$$

The equivalent system of eq. is

$$-5x_1 + x_2 + 3x_3 = 0$$

$$-\frac{4}{5}x_2 + \frac{8}{5}x_3 = 0$$

Let $x_3 = k_2$ be arbitrary

$$-\frac{4}{5}x_2 = -\frac{8}{5}k_2 \Rightarrow \boxed{x_2 = 2k_2}$$

$$-5x_1 + 2k_2 + 3k_2 = 0$$

$$\Rightarrow -5x_1 = -5k_2 \Rightarrow \boxed{x_1 = k_2}$$

$$\therefore \text{Eigen vector is } X_2 = \begin{bmatrix} k_2 \\ 2k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

when $\lambda_3 = 3$:

$$[A - \lambda I] = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 5/2 & 5/2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 3 \\ 0 & 5/2 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p(A') = 2$$

equivalent system of eq. is

$$-2x_1 + x_2 + 3x_3 = 0$$

$$\frac{5}{2}x_2 + \frac{5}{2}x_3 = 0 \Rightarrow x_2 = -x_3$$

let $x_3 = k_3$ be arbitrary

$$x_2 = -k_3$$

$$-2x_1 + 3k_3 - k_3 = 0$$

$$-2x_1 = -2k_3$$

$$x_1 = k_3$$

$$\therefore \text{Eigen vector } X_3 = \begin{bmatrix} k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$3) A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{array}{l} 2(4) - 0 + 1(-2) \\ 8 - 2 = 6 \end{array}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \lambda^3 - 6\lambda^2 + [4 + (4-1) + 4]\lambda - 6 = 0$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = 3, \lambda = 2$$

To find eigen vectors -

(a) when $\lambda_1 = 1$

$$\text{The equivalent matrix is } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A') = 2$$

The equivalent system of eq. is

$$x_1 + x_3 = 0$$

$$x_2 = 0$$

let $x_3 = k_1$ be arbitrary $\Rightarrow x_1 = -k_1$

$$\therefore \text{eigen vector is } x_1 = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} //$$

(b) when $\lambda_2 = 3$,

$$\text{equivalent matrix } [A - \lambda I] = 0 \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A') = 2$$

The equivalent system of eq. is $-x_1 + x_3 = 0$

let $x_3 = k_2$ be arbitrary

$$\Rightarrow -x_1 = -x_3 = k_2$$

$$\therefore \text{Eigen vector is } x_2 = k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} //$$

(c) when $\lambda_3 = 2$

$$\text{equivalent matrix } [A - \lambda I] = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0 \quad \text{let } x_2 = x_3 \text{ arbitrary}$$

$$x_3 = 0$$

$$\therefore \text{Eigen vector is } x_3 = k_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$4) \quad A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$8(21-16) + 6(-18+8) + 2(24-14)$$

$$40 - 60 + 20 = 0$$

$$[A - \lambda I] \neq \emptyset \Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ -2 & -4 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \lambda^3 - 18\lambda^2 + [(21-16) + (24-4) + (56-36)]\lambda - 0 = 0$$

$$= \lambda^3 - 18\lambda^2 + (5+20+20)\lambda - 0 = 0$$

$$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\Rightarrow \lambda_1 = 15, \lambda_2 = 3, \lambda_3 = 0$$

To find eigen vectors -

(a) when $\lambda_1 = 15$

$$\text{equivalent matrix } [A - \lambda I] = 0 \Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' \sim$$

$$x_1 = 1/2$$

$$x_2 = 2$$

$$x_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$1 \quad 2 \quad -2$$

$$1 \quad 1$$

$$0$$

$$5) A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol:- The characteristic equation is $|A - \lambda I| = 0$

$$[A - \lambda I] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$$\Rightarrow \lambda^3 - [-1]\lambda^2 + [-12-3-6]\lambda - 45 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow \lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -5$$

By synthetic division,

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & & -3 & 6 & +45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$$\Rightarrow (\lambda+3)(\lambda^2-2\lambda-15)=0$$

$$\Rightarrow (\lambda+3)(\lambda^2-5\lambda+3\lambda-15)=0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5)=0$$

$$\therefore \lambda = -3, \lambda = 5$$

The eigen values are $\lambda_1 = \lambda_2 = -3, \lambda_3 = 5$

To find Eigen vectors -

(a) when $\lambda = 5$

The equivalent eq. $[A - \lambda I]X = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4 + \frac{4}{7} = \frac{-28+4}{7} = \frac{-24}{7}$$

$$-6 + \frac{6}{7} = \frac{-42+6}{7} = \frac{-36}{7}$$

$$-2 - \frac{2}{7}$$

$$= \frac{-14-2}{7}$$

$$A' \sim \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & -16/7 & -32/7 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{2}{7}R_1$$

$$R_2 \rightarrow R_2 + \frac{2}{7}R_1$$

$$R_3 \rightarrow R_3 - \frac{2}{7}R_2$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{2}{7}R_2$$

$$R_3 \rightarrow R_3 - \frac{2}{7}R_2$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

The equivalent eq. system is

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$-24x_2 - 48x_3 = 0 \Rightarrow -x_2 = 2x_3$$

$$\text{Let } x_3 = k_1 \Rightarrow x_2 = -2k_1$$

$$-7x_1 - 4k_1 - 3k_1 = 0$$

$$-7x_1 = 7k_1$$

$\therefore$ The Eigen vector is

(b) when $\lambda = -3$

The equivalent eq.

$$\begin{bmatrix} -4 & 2 \\ 2 & -4 \\ -1 & -2 \end{bmatrix}$$

$$A' \sim$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_2 \rightarrow R_2 - R_1$$

$$\therefore n - g_i = 3 - 1$$

The equivalent eq.

$$x_1 + 2x_2 = 0$$

$$\text{Let } x_1 = k_1$$

$$3x_3 = 0$$

$\therefore$ The eigen vector

$$-7x_1 - 4k_1 - 3k_1 = 0$$

$$-7x_1 = 7k_1 \Rightarrow x_1 = -k_1$$

$$\therefore \text{The Eigen vector is } X_1 = \begin{bmatrix} -k_1 \\ -2k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

(b) when $\lambda = -3$

The equivalent eq. $[A - \lambda I]X = 0$

$$\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^{-1} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A^{-1}) = 1$$

$R_3 \rightarrow R_3 - R_1$
 $R_2 \rightarrow R_2 - 2R_1$

$\therefore n - \rho = 3 - 1 = 2$ variables are arbitrary.

The equivalent system of eq. is

$$x_1 + 2x_2 - 3x_3 = 0$$

Let $x_1 = k_1'$, $x_2 = k_2'$ be arbitrary.

$$3x_3 = x_1 + 2x_2 \Rightarrow x_3 = \frac{k_1' + 2k_2'}{3}$$

$$\therefore \text{The eigen vector is } X_2 = \begin{bmatrix} x_1 = k_1' \\ x_2 = k_2' \\ x_3 = \frac{k_1' + 2k_2'}{3} \end{bmatrix} = k_1' \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix} + k_2' \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$$

$$\text{or } k_1' \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2' \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

$$6) A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Sol: $[A - \lambda I] = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 5\lambda^2 + [2+4+2]\lambda - 4 = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$$

$$2 \begin{array}{ccc|c} 1 & -5 & 8 & -4 \\ 0 & 2 & -6 & 4 \\ 1 & -3 & 2 & -4 \end{array} \Rightarrow \begin{aligned} &(\lambda - 2)(\lambda^2 - 3\lambda + 2) = 0 \\ &(\lambda - 2)(\lambda^2 - 2\lambda - \lambda + 2) = 0 \\ &(\lambda - 2)(\lambda - 2)(\lambda - 1) = 0 \\ &\therefore \lambda_3 = 2 \end{aligned}$$

To find eigen vectors -

(a) when $\lambda = 1$

equivalent eq. is $[A - \lambda I]X = 0$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A' \sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

$R_3 \leftrightarrow R_1$

$$\sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A') = 2$$

$n - \rho = 3 - 2 = 1$ variable is arbitrary

The equivalent system of eq. is

$$-x_1 + 2x_2 + x_3 = 0$$

$$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$$

Let $x_3 = k_1$

$$\Rightarrow x_2 = -k_1$$

$$-x_1 - 2k_1 + k_1 = 0$$

$$-x_1 = k_1 \Rightarrow x_1 = -k_1$$

The eigen vector is $X_1 = k_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

(b) when $\lambda = 2$, $[A - \lambda I]X = 0$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A' \sim$

$$R_3 \rightarrow R_3 - R_1 \quad \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \rightarrow R_3 + 2R_2 \quad \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A') = 2 \Rightarrow n - \rho = 3 - 2 = 1$ variable be arbitrary

$$-x_1 + 2x_2 + 2x_3 = 0$$

$$x_3 = 0$$

$$\Rightarrow -x_1 + 2x_2 = 0 \Rightarrow x_1 = 2x_2$$

Let $x_2 = k_2$

$$\Rightarrow x_1 = 2k_2$$

$\therefore$ Eigen vector is $X_2 = k_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix} = 2, 3, 5 \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2, 2, 2$$

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$$1(6) - 6(3) + 0$$

$$6 - 18 = -12$$

$$\Rightarrow A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol:- $[A - \lambda I] = \begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 6\lambda^2 + [6+3-4]\lambda + 12 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = 4, \lambda_3 = 3$$

To find eigen vectors -

(a) when $\lambda_1 = -1$

equivalent eq. $[A - \lambda I]X = 0$

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A' \sim \begin{bmatrix} 2 & 6 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow S(A') = \begin{bmatrix} 2 & 6 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - 4R_1$$

Qm *

Solution of Algebraic and Transcendental equations by Newton-Raphson Method:

Intermediate Value Theorem - If a function $f(x)$ is continuous at every point of the interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exist at least one point c b/w a and b for which $f(c) = 0$ $\Rightarrow c$ is the root of the equation $f(x) = 0$

The Newton-Raphson method - The iterative formula is :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Starting with an initial approximation $x_0 = \frac{a+b}{2}$ or given value,

x_1 = 1st approximate value.

x_2 = 2nd approx. value

!

This is continued until 2 iterated values are same upto 4 decimals.

1) Solve the equation $3x = \cos x + 1$ in the interval $(0, 1)$ upto 4 decimals.

Sol: $f(x) = 3x - \cos x - 1 = 0$
 $\Rightarrow f'(x) = 3 + \sin x$

$f(0) = -2$ (-ve)

$f(1) = 1.45$ (+ve)

$f(0.5) = -0.37 \rightarrow$ approx. b/w 0.5

Let $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{(-0.377)}{3.4794} = 0.5 + \frac{0.377}{3.4794} = 0.6083 //$$

$\log \rightarrow \text{base } 10$
 $\ln \rightarrow \text{base } e$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{0.6083 - \frac{0.0042}{3.5714}}{}$$

$$x_2 = 0.6071 //$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071 + 0$$

$$x_3 = 0.6071 //$$

$\therefore$ The root is $0.6071 //$

2) $f(x) = x^3 - 3x + 1 = 0$ near $x = 0.5$

Sol: $f'(x) = 3x^2 - 3$ $f(0.5) = -0.3750$
 Let $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{0.5 - (0.375)}{2.250}$$

$$\Rightarrow x_1 = 0.3333 //$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{0.3333 + \frac{0.0371}{2.6667}}{}$$

$$\Rightarrow x_2 = 0.3472 //$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{0.3472 + \frac{0.0002}{2.6383}}{}$$

$$\Rightarrow x_3 = 0.3472 //$$

$\therefore$ The root is $0.3472 //$

* 3) Find the real root of $x \log_{10} x = 1.2$ in $(2, 3)$ correct to 4 decimal places.

Sol: $f(x) = x \log_{10} x - 1.2 = 0 \Rightarrow f(x) = x \frac{\log_e x}{\log_e 10} - 1.2$
 $\log_e 10 \rightarrow \frac{1}{\log_e 10} = \log_{10} e$

$$\Rightarrow f(x) = 0.4343 x \log_e x - 1.2 = 0$$

$$f'(x) = 0.4343 \left[x \left(\frac{1}{x} \right) + \log_e x (1) \right]$$

$$f'(x) = 0.4343 [1 + \log_e x]$$

Now let $x_0 = 2.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{(-0.2051)}{0.8322}$$

$$x_1 = 2.7464 //$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7464 - \frac{0.0050}{0.8730}$$

$$x_2 = 2.7406 //$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7406 + 0$$

$$x_3 = 2.7406 //$$

$\therefore$ The root is $2.7406 //$

4) $x e^x - 2 = 0$ near $x = 0.5$

Sol:-

$$f(x) = x e^x - 2 = 0$$

$$f'(x) = x e^x + e^x$$

For $x_0 = 0.5$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 + \frac{1.1756}{2.4730}$$

$$x_1 = 0.9753 //$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.9753 - \frac{0.5866}{5.2384}$$

$$x_2 = 0.8633 //$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{0.8633 - 0.0469}{4.4178}$$

$$x_3 = 0.8526 //$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{0.8526 + 0}{0}$$

$$x_4 = 0.8526 //$$

$\therefore$ The root is $0.8526 //$

5) $f(x) = x \sin x + \cos x = 0$ near $x = \pi$

Sol: $f'(x) = x \cos x + \sin x - \sin x \neq 0$
 $= x \cos x = 0$

For $x_0 = \pi$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{1}{\pi} = 2.8232 //$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{2.8232 - 0.0659}{2.6813}$$

$$x_2 = 2.7986 //$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{2.7986 - 0.0005}{2.6856} = 2.7984 //$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{2.7984 - 0}{0} = 2.7984 //$$

$\therefore$ The root is $2.7984 //$

(6) Find the cube root of 24 correct to 3 decimals.

Sol: Let $x = \sqrt[3]{24}$

$$\Rightarrow x^3 = 24 \Rightarrow x^3 - 24 = 0$$

$$f(x) = x^3 - 24 = 0$$

$$f'(x) = 3x^2$$

For $x_0 = 3$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3}{27} = 2.8888 //$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8888 - \frac{0.1075}{25.035} = 2.8845 //$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.8845 - 0 = 2.8845 //$$

$\therefore$ The cube root is 2.8845 //

* Finite Differences -

Suppose x values are equally spaced i.e. $x_0, x_1, x_2, \dots, x_n$ ($x_n = x_0$) and the corresponding y values are $y_0, y_1, \dots, y_n$.

The forward difference denoted by ' Δ ' is given as,

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\vdots$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The second forward difference meaning $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 \dots \dots$

The backward difference denoted by ' ∇ ' is given as,

∇y_0 does not exist

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$\vdots$$

$$\nabla y_n = y_n - y_{n-1}$$

The second backward difference i.e. $\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$

- 1) Construct a forward difference table for the function $f(x) = x^3 + 5x = 7$ for $x = 1, 2, 3, 4, 5$.

Sol:

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1	-1	12			
2	11	24	12		
3	35	42	18	6	
4	77	66	24	6	0
5	143				

leading forward difference - $\Delta y_0 = 12, \Delta^2 y_0 = 12, \Delta^3 y_0 = 6, \Delta^4 y_0 = 0$

leading backward differences

$$\Rightarrow \nabla y_4 = 66, \nabla^2 y_4 = 24, \nabla^3 y_4 = 6, \nabla^4 y_4 = 0$$

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* INTERPOLATION -

For a given set of values of x & y , the process of finding an approximate value of y b/w 2 values of x is called Interpolation.

(Extrapolation $[x_0, x_n]$ - outside the range)

Interpolation for equal intervals:

- ① Newton's forward interpolation formula (Newton-Gregory formula) - given by,

$$y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots$$

where $x = x_0 + ph$ $h \rightarrow$ step length; $x_0 \rightarrow$ first term of x

$$p = \frac{x - x_0}{h}$$

NOTE:

This formula is used when the value of x is near the beginning of the table.

② Newton's Backward Difference formula (Newton-Gregory backward diff. formula) - given as,

$$y(x) = y_n + r \nabla y_n + \frac{r(r+1)}{2!} \nabla^2 y_n + \frac{r(r+1)(r+2)}{3!} \nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 y_n + \dots$$

where $x = x_n + rh$ x_n is the last term of x

$$r = \frac{x - x_n}{h}$$

NOTE:

This formula is used when the value of x is near the end of the table.

1) Given the table,

Temperature	140	150	160	170	180
Pressure	3.685	4.854	6.302	8.076	10.225

compute the pressure of steam at temp. 142°

Sol:

$x = \text{temp}$	$y = \text{pressure}$	Δ	Δ^2	Δ^3	Δ^4
140	3.685	1.169			
150	4.854	1.448	0.279	0.047	0.002
160	6.302	1.774	0.326	0.049	
170	8.076	2.149	0.375		
180	10.225				

$$x = 142$$

$$\therefore y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$p = \frac{x - x_0}{h} = \frac{142 - 140}{10} = \frac{2}{10} = 0.2$$

$$y(x) = 3.685 + 0.2(1.169) + \frac{0.2(0.2-1)}{2} \cdot 0.279 + \frac{0.2(0.2-1)(0.2-2)}{6} \cdot 0.047 + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{24} \cdot (0.002)$$

$$= 3.685 + 0.2338 - 0.0223 + 0.0022 - 0$$

$$y(x) = 3.898$$

$$x = 175$$

$$y(x) = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_n$$

$$p = \frac{x - x_n}{h} = \frac{175 - 180}{10} = \frac{-5}{10} = -0.5$$

$$y(x) = 10.225 + (-0.5)(2.149) + \frac{(-0.5)(-0.5+1)}{2} \cdot 0.375 + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6} \cdot 0.049 + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24} \cdot (0.002)$$

$$= 10.225 - 1.0745 - 0.375 - 0.0030 - 0$$

$$= 8.7725$$

2)	x	y	Δ	Δ^2	Δ^3
	40	184	20		
	50	204	22	2	
	60	226	24	2	0
	70	250	26	2	0
	80	276	28	2	0
	90	304			

$$x = 38$$

$$y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0$$

$$p = \frac{x - x_0}{h} = \frac{38 - 40}{10} = -0.2$$

$$\therefore y(x) = 184 + (-0.2)(20) + \frac{(-0.2)(-0.2-1)}{2} (2) \quad (2)$$

$$= 184 - 4 + 0.24$$

$$y(x) = 180.24 //$$

$$x = 85$$

$$y(x) = y_n + p \nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n$$

$$p = \frac{x - x_n}{h} = \frac{85 - 90}{10} = -0.5$$

$$\therefore y(x) = 304 + (-0.5)(28) + \frac{(-0.5)(-0.5+1)}{2} (2) \quad (2)$$

$$= 304 - 14 - 0.25$$

$$= 289.75 //$$

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- 3) The following table gives the details about the parameters x & y as follows :

x	200	250	300	350	400
y	15.04	16.81	18.42	19.9	21.27

Find the value of y at $x = 218$ using appropriate formula.

Sol:-

x	y	Δ	Δ^2	Δ^3	Δ^4
200	15.04	1.77			
250	16.81	1.61	-0.16	0.03	-0.01
300	18.42	1.48	-0.13	0.02	
350	19.9	1.37	-0.11		
400	21.27				

$$x = 218$$

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$p = \frac{x - x_0}{h} = \frac{218 - 200}{50} = 0.36$$

$$\begin{aligned} y(x) &= 15.04 + (0.36)(1.77) + \frac{(0.36)(0.36-1)}{2} (-0.16) + \frac{(0.36)(0.36-1)}{6} \\ &\quad \frac{(0.36-2)}{24} (0.03) + \frac{(0.36)(0.36-1)(0.36-2)(0.36-3)}{24} (-0.01) \\ &= 15.04 + 0.6372 + 0.0184 + 0.0018 + 0.0004 \end{aligned}$$

$$y(x) = 15.697$$

4)	$x = \text{diameter (D)}$	$y = \text{area (A)}$	Δ	Δ^2	Δ^3	Δ^4
	80	5026	648			
	85	5674	688	40		
	90	6362	726	38	-2	4
	95	7088	766	40	2	
	100	7854				

$$x = 105$$

$$h = \frac{x - x_n}{h} = \frac{105 - 100}{5} = 1$$

$$\therefore y(x) = y_n + h \nabla y_n + \frac{h(h+1)}{2!} \nabla^2 y_n + \frac{h(h+1)(h+2)}{3!} \nabla^3 y_n +$$

$$+ \frac{h(h+1)(h+2)(h+3)}{4!} \nabla^4 y_n$$

$$= 7854 + (1)(766) + \frac{(1)(2)}{2} (40) + \frac{(1)(2)(3)}{6} (2)$$

$$+ \frac{(1)(2)(3)(4)}{24} (4)$$

$$y(x) = 8666 //$$

5) Find an interpolating polynomial $f(x)$ which satisfies the given data:
 $f(0)=0$, $f(2)=4$, $f(4)=56$, $f(6)=204$, $f(8)=496$, $f(10)=980$
 ~~$f(2)=4$~~

Sol:

x	y	Δ	Δ^2	Δ^3	Δ^4
0	0	4			
2	56 4		48		
4	204 56	52		48	
6	496 204	148	96		0
8	496	292	144	48	0
10	980	484	192		

$$p = \frac{x-x_0}{h} \Rightarrow \boxed{p = \frac{x}{2}}$$

$$y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

$$= 0 + \frac{x}{2} (4) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)}{2} (48) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6} (48)$$

$$\frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6} = -\frac{x^3}{8} + 2x^2 - 2x$$

$$= 0 + 2x + (-6x^2) + 3x^3 - 12x + 8x^2 + 8x$$

$$\frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6} = \frac{3x^3 - 12x^2 + 8x}{8} \quad y(x) = x^3 - 2x //$$

$$\frac{(x-2)x}{2} = \frac{x^2 - 2x}{2} = \frac{1}{2}x^2 - x$$

$$\frac{x(x-2)(x-4)}{2} = \frac{(x^2 - 2x)(x-4)}{2} = \frac{x^3 - 2x^2 - 4x^2 + 8x}{2} = \frac{x^3 - 6x^2 + 8x}{2}$$

$$\frac{x(x-2)(x-4)}{2} = \frac{x^3 - 6x^2 + 8x}{2}$$

6) Given: $f(-4) = -25$, $f(-2) = 1$, $f(0) = 3$, $f(2) = 29$, $f(4) = 127$

find $f(x)$ using Newton Backward formula.

Sol:

x	y	Δ	Δ^2	Δ^3	Δ^4
-4	-25				
-2	1	26	-24	48	0
0	3	2	24	48	
2	29	26	72		
4	127	98			

$$x = x_n + nh$$

$$\Rightarrow n = \frac{x - x_n}{h} = \frac{x - 4}{2} = \left(\frac{x - 2}{2}\right)$$

$$\therefore y(x) = y_n + n \nabla y_n + \frac{n(n+1)}{2!} \nabla^2 y_n + \frac{n(n+1)(n+2)}{3!} \nabla^3 y_n$$

$$y(x) = 127 + \frac{x-4}{2} (98) + \left(\frac{x-4}{2}\right) \left(\frac{x-2+1}{2}\right) \frac{(72)}{2} + \left(\frac{x-4}{2}\right)$$

$$\left(\frac{x-2+1}{2}\right) \left(\frac{x-2+2}{2}\right) \frac{1}{6} (48)$$

$$= 127 + 49x - 196 + \left(\frac{x-4}{2}\right) \left(\frac{x-2}{2}\right) (36) + \left(\frac{x-4}{2}\right) \left(\frac{x-2}{2}\right)$$

$$\left(\frac{x}{2}\right) (8)$$

$$= 127 - 196 + 49x + (x^2 - 4x - 2x + 8) (4) +$$

$$(x-4)(x-2)(x)$$

$$= 127 - 196 + 49x + 9x^2 - 54x + 72 + (x^2 - 4x)(x-2)$$

$$= 127 - 196 + 49x + 9x^2 - 54x + 72 + x^3 - 4x^2 - 2x^2 + 8x$$

$$y(x) = x^3 + 3x^2 + 3x + 3 //$$

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- * 7) From the given table, find the no. of students who have obtained
- less than 45 marks
 - blw 40 & 45 marks

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Sol:- Let x be the marks and y be the no. of students having less than x marks

x	y	Δ	Δ^2	Δ^3	Δ^4
40	31	42			
50	$(42+31)=73$	51	9		
60	$(73+51)=124$	35	-16	-25	37
70	$(124+35)=159$	31	-4	12	
80	$(159+31)=190$				

a) $x = 45$

$$p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = \frac{5}{10} = 0.5$$

$$y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 +$$

$$\frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 31 + (0.5) 42 + \frac{(0.5)(0.5-1)}{2} (9) + \frac{(0.5)(0.5-1)(0.5-2)}{6} (-25)$$

$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} (37)$$

$$= 31 + \frac{21}{2} + (-1.125) + (-1.5625) + (-1.4453)$$

$$y(x) = 47.86 \approx 48$$

No. of students having marks less than 45 ≈ 48

b) No. of students having marks between 40 and 45 = 48 - (no. of students having marks < 40)

$$= 48 - 31 = 17$$

8) If the no. of persons earning below ₹1000 in a slum locality is 6000. Estimate the no. of persons having income blw ₹2000 and ₹2500 from the given table.

Income	1000-2000	2000-3000	3000-4000	4000-5000
No. of persons	4250	3600	1500	650

Sol:-

Let x be the income then y be the no. of persons having income less than x .

x	y	Δ	Δ^2	Δ^3
2000	$(4250 + 6000) = 10250$	3600	-2100	1250
3000	13850	1500	-850	
4000	15350	650		
5000	16000			

$$x = 2500$$

$$p = \frac{x - x_0}{h} = \frac{2500 - 2000}{1000} = \frac{500}{1000} = 0.5$$

$$y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

$$y(x) = 10250 + (0.5)(3600) + \frac{(0.5)(0.5-1)}{2} (-2100) + \frac{0.5(0.5-1)(0.5-2)}{6} (1250)$$

$$= 10250 + 1800 + 262.5 + 78.125$$

$$y(x) = 12390.625$$

$$\approx 12391 \quad (< 2500)$$

$$\text{No. of persons having income blw 2000 \& 2500} = 12391 - 10250$$

$$\approx 2141 //$$

9) If x is the no. of points scored by a person in a competition, y is the total no. of persons scoring x points, from the given table, find the no. of persons who score points b/w 20 and 25.

x	under 10	10-20	20-30	30-40	40-50
y	20	45	115	210	115

Sol:-

x	y	Δ	Δ^2	Δ^3	Δ^4
10	20	45			
20	65	115	70		
30	180	210	95	25	
40	390	115	-95	-190	-215
50	505				

$$x = 25$$

$$p = \frac{x - x_0}{h} = \frac{25 - 10}{10} = 1.5 //$$

$$y(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 20 + 1.5(45) + \frac{(1.5)(1.5-1)}{2!} (70) + \frac{(1.5)(1.5-1)(1.5-2)}{3!} (25) + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{4!} (-215)$$

$$= 20 + 67.5 + 26.25 + (-1.5625) + \frac{5.0390}{24}$$

$$= 107.14$$

$$\approx 107 //$$

$$\text{No. of persons who score points b/w 20 \& 25} = 107 - 65$$

$$\approx 42 //$$

10) Marks 0-20 20-40 40-60 60-80 80-100

no. of students 41 62 65 50 17

Find the no. of students who have scored less than 70 marks

Sol: let x be the marks and y be the no. of students scoring less than x .

x	y	Δ	Δ^2	Δ^3	Δ^4
20	41	62			
40	$(62+41)=103$	65	3	-18	0
60	168	50	-15	18	
80	218	17	-33		
100	235				

$$x = 70$$

$$h = \frac{x - x_n}{h} = \frac{70 - 100}{20} = -1.5$$

$$y(x) = y_n + h \nabla y_n + \frac{h(h+1)}{2!} \nabla^2 y_n + \frac{h(h+1)(h+2)}{3!} \nabla^3 y_n + \frac{h(h+1)(h+2)(h+3)}{4!} \nabla^4 y_n$$

$$y(x) = 235 + (-1.5)(17) + \frac{(-1.5)(-1.5+1)(-33)}{2!} + \frac{(-1.5)(-1.5+1)(-1.5+2)(-18)}{3!} + 0$$

$$= 235 - 25.5 - \frac{24.75}{2} - \frac{6.75}{6}$$

$$= 235 - 25.5 - 12.375 - 1.125$$

$$y(x) = 196$$

* Lagrange's Interpolation Formula -

It is used for unequal intervals.

If $y = f(x)$ takes the values $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ then, the Lagrange's formula for finding y at x is given as:

$$y(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} \times y_1$$

$$+ \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} \times y_n$$

Lagrange's Inverse interpolation is used to find the value of x for a particular y and the Lagrange's formula for finding y at x is given as:

$$x(y) = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_n)} \times x_0 + \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)} \times x_1$$

$$+ \dots + \frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})} \times x_n$$

- 1) The following are the measurements T made on a curve recorded by oscillograph representing a change of current I due to a change in the conditions of an electric current. Find I at $T=1.6$ using appropriate formula.

T	1.2	2.0	2.5	3.0
I	1.36	0.58	0.34	0.20

Sol:- $y \Rightarrow I, x \Rightarrow T$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = \frac{(1.6-2.5)(1.6-3.0)(1.6-3.0)}{(1.2-2.5)(1.2-3.0)(1.2-3.0)} (1.36) + \frac{(1.6-1.2)(1.6-3.0)(1.6-3.0)}{(2.0-1.2)(2.0-3.0)(2.0-3.0)} (0.58)$$

$$+ \frac{(0.4)(-0.4)(-1.4)(0.34)}{(1.3)(0.5)(-0.5)} + \frac{(0.4)(-0.4)(-0.9)(0.2)}{(1.8)(1)(0.5)}$$

$$y = 0.3661 + 0.7308 - 0.2343 + 0.0320$$

$$y = 0.8946 //$$

- 2) The observed values of a function are respectively 168, 120, 72 and 63 at 4 positions 3, 7, 9, 10 of the independent variable. What is the best estimate for the value of the function at the position 6 of the independent variable?

Sol:-

x	3	7	9	10
y	168	120	72	63

at $x=6$

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y(x) = \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)} (168) + \frac{(3)(-3)(-4)}{4(-2)(-3)} (120) + \frac{(3)(-1)(-4)}{(6)(2)(-1)} (72)$$

$$+ \frac{(3)(-1)(-3)}{(7)(3)(1)} (63)$$

$$y(x) = 12 + 180 - 72 + 27$$

$$= 147 //$$

3) Given:

x	20	30	40
y	2	4.4	7.9

Find x when $y=6$

Sol:-

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$x = \frac{(1.6)(-1.9)(20)}{(-2.9)(-5.9)} + \frac{(4)(-1.9)(30)}{(2.4)(-3.5)} + \frac{(4)(1.6)(40)}{(5.9)(3.5)}$$

$$x = -4.2937 + 27.1428 + 12.3970$$

$$x = 35.2461 //$$

- 4) Apply Lagrange's formula inversely to find the root of the equation $f(x)=0$ given that it satisfies $f(30)=-30$, $f(34)=-13$, $f(38)=3$, $f(42)=18$.

Sol:-

x	30	34	38	42
y	-30	-13	3	18

given $f(x)=0 \Rightarrow y=0$

Root of the eq.ⁿ $\Rightarrow$ finding the value of x for which y is 0.

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} x_1 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$$x = \frac{(13)(-3)(-18)}{(-17)(-33)(-48)} (30) + \frac{(30)(-3)(-18)}{(17)(-16)(-31)} (34) + \frac{(30)(13)(-18)}{(33)(16)(-15)}$$

$$= (38) + \frac{(30)(13)(-3)}{(48)(31)(15)} (42)$$

$$x = 0.7820 + 6.5322 + 33.6818 - 2.2016$$

$$x = 37.2303 //$$

- 5) Find the interpolating polynomial of the form $x=f(y)$ and hence find the value of $x(5)$ and $y(5)$.

x	2	10	17
y	1	3	4

Sol:- $x = f(y) \Rightarrow$ Inverse Lagrange's formula.

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$= \frac{(y-3)(y-4)}{(-2)(-3)} (2) + \frac{(y-1)(y-4)}{(2)(-1)} (10) + \frac{(y-1)(y-3)}{(3)(1)} (17)$$

$$= \frac{1}{3} (y^2 - 3y - 4y + 12) - 5(y^2 - y - 4y + 4) + \frac{17}{3} (y^2 - y - 3y + 3)$$

$$= \frac{1}{3} (y^2 - 7y + 12) - 5(y^2 - 5y + 4) + \frac{17}{3} (y^2 - 4y + 3)$$

$$= \frac{1}{3} y^2 - 5y^2 + \frac{17}{3} y^2 - \frac{7}{3} y + 25y - \frac{17}{3} (4y) + 4 - 20 + 17$$

$$x = y^2 + 0 + 1$$

$$\boxed{x = y^2 + 1} \text{ ——— ①}$$

$$\text{at } y=5 \Rightarrow x(5) = 25 + 1 = 26 //$$

$$\text{for } y(5) \Rightarrow \text{put } x=5 \text{ in eq ①,}$$

$$5 = y^2 + 1$$

$$y^2 = 4$$

$$\Rightarrow y = \pm 2 //$$

- 6) A curve passes through the following points $(0, 18)$ $(1, 10)$ $(3, -18)$ $(6, 90)$. Find the slope of the curve at $x=2$.

Sol:- Get the curve $y = f(x) \Rightarrow \therefore$ Slope of the curve $= \frac{dy}{dx} = f'(x)$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{x=2} = \text{slope of curve at } x=2$$

$$x \quad 0 \quad 1 \quad 3 \quad 6$$

$$y \quad 18 \quad 10 \quad -18 \quad 90$$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = \frac{(x-1)(x-3)(x-6)}{(-1)(-3)(-6)} (18) + \frac{(x-0)(x-3)(x-6)}{(1)(-2)(-5)} (10) +$$

$$\frac{(x)(x-1)(x-6)}{(3)(2)(-3)} (-18) + \frac{x(x-1)(x-3)}{(6)(5)(3)} (90)$$

$$\Rightarrow y = -\frac{(x-1)(x-3)(x-6)}{6} - \frac{x(x-3)(x-6)}{10} + \frac{x(x-1)(x-6)}{6} + \frac{x(x-1)(x-3)}{10}$$

$$\Rightarrow y = -\frac{(x^2 - x - 3x + 3)(x-6)}{6} - \frac{(x^2 - 3x)(x-6)}{10} + \frac{(x^2 - x)(x-6)}{6} + \frac{(x^2 - x)(x-3)}{10}$$

$$\Rightarrow y = -\frac{(x^3 - x^2 - 3x^2 + 3x - 6x^2 + 6x + 18x - 18)}{6} - \frac{(x^3 - 3x^2 - 6x^2 + 18x)}{10} + \frac{(x^3 - x^2 - 6x^2 + 6x)}{6} + \frac{(x^3 - x^2 - 3x^2 + 3x)}{10}$$

$$\Rightarrow y = -\frac{x^3 + x^2 + 3x^2 - 3x + 6x^2 - 6x - 18x + 18}{6} - \frac{x^3 + 3x^2 + 6x^2 + 18x}{10} + \frac{x^3 - x^2 - 3x^2 + 3x}{6} + \frac{x^3 - x^2 - 3x^2 + 3x}{10}$$

$$\Rightarrow y = 2x^3 - 10x^2 + 18$$

$$\Rightarrow \frac{dy}{dx} = 6x^2 - 20x \Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 24 - 40 = -16 //$$

7) The velocity vs time data is given below

t 0 1 3

v 21 16 12

Find the distance moved by the particle and its acceleration at the end of 4 seconds.

Sol:- $a = \frac{dv}{dt}$; $f(x) = y \Rightarrow a$ at $t=4$

$$\text{distance } s = \int \frac{ds}{dt} dt = \int v dt = \int y dx$$

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$y = \frac{(x-1)(x-3)}{(-1)(-3)} (21) + \frac{(x-0)(x-3)}{(1)(-2)} (16) + \frac{(x-0)(x-1)}{(-3)(-2)} (12)$$

$$y = \frac{(x-1)(x-3)}{3} (21) + -\frac{(x^2-3x)(8)}{2} + \frac{(x^2-x)2}{2}$$

$$= (x^2 - x - 3x + 3)7 - 8x^2 + 24x + 2x^2 - 2x$$

$$= 7x^2 - 8x^2 - 28x + 4x - 2x + 21$$

$$y = -x^2 - 26x + 21$$

$$a = \frac{dv}{dt} = -2x - 26$$

at

$$\text{at } x=4 \Rightarrow a = -2(4) - 26$$

$$a = -34$$

$$s = \int v dt = \int_0^4 -x^2 - 26x + 21 = \left[-\frac{x^3}{3} - \frac{26x^2}{2} + 21x \right]_0^4$$

$$s = \frac{-64}{3} - 13(8) + 21(4)$$

$$s = -41.33$$

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 $a = 2$ $b = 57.33$ 

* Numerical Integration

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called Numerical Integration (Quadrature).

$$\text{Let } I = \int_a^b f(x) dx$$

Let $x_0 = a$ and $x_n = b$ such that h (step size) is $\frac{b-a}{n}$ where

n is the number of equal parts.

Therefore,

(1) Trapezoidal Rule -

$$I = \int_{x_0}^{x_0+nh} f(x) dx = \int_{x_0}^{x_0+nh} y dx = \frac{h}{2} \left[(y_0 + y_n) + 2(\text{Remaining values of } y) \right]$$

$$= \frac{h}{2} \left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}) \right]$$

Trapezium rule is used for any value of n .

(2) Simpson's $\frac{1}{3}$ rd Rule -

$$I = \frac{h}{3} \left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots) \right]$$

$$= \frac{h}{3} \left[(y_0 + y_n) + 4(\text{Odd terms of } y) + 2(\text{Even terms of } y) \right]$$

This rule is used only when n is an even number.

NOTE:

When an interval (a, b) is divided into n equal parts, we get $n+1$ values of x and their corresponding values of y .

Therefore, n equal parts implies we have $n+1$ ordinates.

Evaluate the given integrals

1) $\int_0^6 \frac{dx}{1+x^2}$ by taking 6 equal parts.

Sol:- $f(x) = \frac{1}{1+x^2}$; $h = \frac{b-a}{n} = \frac{6-0}{6} = 1 //$

x	0	1	2	3	4	5	6
$y=f(x)$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

By Trapezium Rule,

$$I = \frac{h}{2} \left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}) \right]$$

$$I = \frac{1}{2} \left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5) \right]$$

$$= \frac{1}{2} \left[\left(1 + \frac{1}{37}\right) + 2\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26}\right) \right]$$

$$= \frac{1}{2} \left[1.0270 + 1.7945 \right]$$

$$I = 1.4107 //$$

By Simpson's Rule,

$$I = \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$= \frac{1}{3} \left[\left(1 + \frac{1}{37}\right) + 4\left(\frac{1}{2} + \frac{1}{10} + \frac{1}{26}\right) + 2\left(\frac{1}{5} + \frac{1}{17}\right) \right]$$

$$= \frac{1}{3} \left[1.0270 + 2.5538 + 0.5176 \right]$$

$$I = 1.3661 //$$

2) $\int_0^1 \frac{dx}{1+x^2}$ by taking 4 equal interval & hence evaluate approx. value of π .

Sol:- $f(x) = \frac{1}{1+x^2}$; $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y=f(x)$	1	$\frac{16}{25}$ $\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$

By Trapezium Rule,

$$I = \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{4} \left(\frac{1}{2} \right) \left[(1 + \frac{1}{2}) + 2 \left(\frac{16}{17} + \frac{4}{5} + \frac{16}{25} \right) \right]$$

$$= \frac{1}{8} [1.5 + \frac{2.3811}{4.7623}]$$

$$I = \frac{0.485}{0.7827} //$$

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{1}{4 \times 3} \left[(1 + \frac{1}{2}) + 4 \left(\frac{16}{17} + \frac{16}{25} \right) + 2 \left(\frac{4}{5} \right) \right]$$

$$= \frac{1}{12} [1.5 + 6.3247 + 1.6]$$

$$I = 0.7853 //$$

Actual Integration : $I = \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}(x)]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$$I = \frac{\pi}{4}$$

$$\Rightarrow \boxed{\pi \approx 4I}$$

Evaluate the given integrals

1) $\int_0^6 \frac{dx}{1+x^2}$ by taking 6 equal parts.

Sol:- $f(x) = \frac{1}{1+x^2}$; $h = \frac{b-a}{n} = \frac{6-0}{6} = 1 //$

x	0	1	2	3	4	5	6
$y=f(x)$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$	$\frac{1}{26}$	$\frac{1}{37}$

By Trapezium Rule,

$$I = \frac{h}{2} \left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1}) \right]$$

$$I = \frac{1}{2} \left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5) \right]$$

$$= \frac{1}{2} \left[\left(1 + \frac{1}{37}\right) + 2\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26}\right) \right]$$

$$= \frac{1}{2} \left[1.0270 + 1.7945 \right]$$

$$I = 1.4107 //$$

By Simpson's Rule,

$$I = \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$= \frac{1}{3} \left[\left(1 + \frac{1}{37}\right) + 4\left(\frac{1}{2} + \frac{1}{10} + \frac{1}{26}\right) + 2\left(\frac{1}{5} + \frac{1}{17}\right) \right]$$

$$= \frac{1}{3} \left[1.0270 + 2.5538 + 0.5176 \right]$$

$$I = 1.3661 //$$

2) $\int_0^1 \frac{dx}{1+x^2}$ by taking 4 equal interval & hence evaluate approx. value of π .

Sol:- $f(x) = \frac{1}{1+x^2}$; $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y=f(x)$	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$

By Trapezium Rule,

$$I = \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{4} \left(\frac{1}{2} \right) \left[\left(1 + \frac{1}{2} \right) + 2 \left(\frac{16}{17} + \frac{4}{5} + \frac{16}{25} \right) \right]$$

$$= \frac{1}{8} [1.5 + \frac{2.384}{4.7623}]$$

$$I = \frac{0.485}{0.7827} //$$

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{1}{4 \times 3} \left[\left(1 + \frac{1}{2} \right) + 4 \left(\frac{16}{17} + \frac{16}{25} \right) + 2 \left(\frac{4}{5} \right) \right]$$

$$= \frac{1}{12} [1.5 + 6.3247 + 1.6]$$

$$I = 0.7853 //$$

Actual Integration : $I = \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}(x)]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$$I = \frac{\pi}{4}$$

$$\Rightarrow \boxed{\pi \approx 4I}$$

Case (1) - $\pi \approx 4(0.7827)$ {by Trapezium Rule}
 $\pi \approx 3.1308 //$

Case (2) - $\pi \approx 4(0.7853)$ {by Simpson's Rule}
 $\pi \approx 3.1412 //$

3) Evaluate $\int_0^1 \frac{dx}{1+x}$ by taking 7 ordinates using Simpson's Rule and hence

deduce an approx. value of $\log_e 2$.

Sol:- $f(x) = \frac{1}{1+x}$; $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} //$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$f(x) = y$	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{6}{11}$	$\frac{1}{2}$

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{6 \times 3} \left[\left(\frac{1}{1} + \frac{1}{2} \right) + 4 \left(\frac{6}{7} + \frac{2}{3} + \frac{6}{11} \right) + 2 \left(\frac{3}{4} + \frac{3}{5} \right) \right]$$

$$= \frac{1}{18} [1.5 + 8.2770 + 2.7]$$

$$I = 0.6931 //$$

Actual integration : $I = \int_0^1 \frac{dx}{1+x} = [\log_e(1+x)]_0^1 = (\log_e 2 - \log_e 1)$

$$I \approx \log_e 2$$

$\therefore$ By Simpson's Rule,

$$\log_e 2 \approx 0.6931 //$$

4) $\int_0^{\pi/2} \sqrt{\cos \theta} d\theta$ taking $n=6$ using Simpson's Rule.

Sol:- $f(x) = \sqrt{\cos \theta}$; $h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
y	1	0.9828	0.9306	0.8408	0.7071	0.5087	0

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{\pi}{12 \times 3} [(1+0) + 4(0.9828 + 0.8408 + 0.5087) + 2(0.9306 + 0.7071)]$$

$$= \frac{\pi}{36} (1 + 9.3292 + 3.2754)$$

$$I = 1.1872 //$$

5) $\int_0^{\pi/2} \sqrt{\cos \theta} d\theta$ taking $n=7$

Sol:- $h = \frac{b-a}{n} = \frac{\pi}{14}$; $f(x) = \sqrt{\cos \theta}$

x	0	$\pi/14$	$2\pi/14$	$3\pi/14$	$4\pi/14$	$5\pi/14$	$6\pi/14$	$\pi/2$
y	1	0.9873	0.9491	0.8842	0.7896	0.6586	0.4717	0

By Trapezium Rule,

$$I = \frac{h}{2} [(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$= \frac{\pi}{14 \times 2} [(1+0) + 2(0.9873 + 0.9491 + 0.8842 + 0.7896 + 0.6586 + 0.4717)]$$

$$= \frac{\pi}{28} (1 + 9.4810) = \frac{1.1759}{1.0637} //$$

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$$\text{Find } \int_4^{5.2} \log x \, dx$$

6) Given

x	$\log x$
4	1.3863
4.2	1.4351
4.4	1.4516
4.6	1.5261
4.8	1.5686
5.0	1.6094
5.2	1.6484

by (a) Trapezium Rule
(b) Simpson's $\frac{1}{3}$ rd Rule

Sol:- $f(x) = \log x$; $h = \frac{5.2-4}{6} = \frac{1}{5}$

(a) By Trapezium Rule,

$$I = \frac{h}{2} \left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5) \right]$$

$$= \frac{1}{10} \left[(1.3863 + 1.6484) + 2(1.4351 + 1.4516 + 1.5261 + 1.5686 + 1.6094) \right]$$

$$= \frac{1}{10} (3.0347 + 15.1816)$$

$$= 1.8216 \text{ // } 1.8236$$

(b) By Simpson's Rule,

$$I = \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right]$$

$$= \frac{1}{15} \left[(1.3863 + 1.6484) + 4(1.4351 + 1.5261 + 1.6094) + 2(1.4516 + 1.5686) \right]$$

$$= \frac{1}{15} [3.0347 + 18.2824 + 6.0404]$$

$$= 1.8238 \text{ // }$$

7) A curve is drawn to pass through the points as given in the table. Estimate the area bounded by the curve, the x-axis & the lines $x=1$ and $x=4$.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Sol:- Required Area = $\int_{x=1}^4 y \, dx$

$f(x) = y$; $h = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{6} [4.1 + \overset{31.2}{\cancel{21.2}} + 11.4]$$

$I = 7.7833$

contd:6) Actual integration : $I = \int_1^{5.2} \log x \, dx = \left[\frac{1}{x} \right]_1^{5.2} = 1.8278$

8) The velocity v of a bike which starts from rest is given at fixed intervals of time as follows:

$t(\text{min})$	2	4	6	8	10	12	14	16
$v(\text{km/min})$	10	18	25	29	32	20	11	5
$t \rightarrow$	18	20						
$v \rightarrow$	2	0						

Estimate the distance covered in 20 minutes.

Sol:- starts from rest $\Rightarrow$ At $t=0$ $v=0$
 $\Rightarrow x_0=0$ $y_0=0$

$$v = \frac{ds}{dt}$$

$$ds = v dt$$

$$\Rightarrow s = \int v dt = \int_0^{20} v dt$$

$\therefore n = 10$ (11 ordinates)

$$f(x) = v \quad ; \quad h = \frac{20-0}{10} = 2$$

By Simpson's Rule,

$$I = \frac{h}{3} \left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8) \right]$$

$$= \frac{2}{3} \left[(0+0) + 4(10+25+32+11+2) + 2(18+29+20+5) \right]$$

$$= \frac{2}{3} [320 + 144]$$

$$I = 309.33 \text{ km}$$

- 9) The velocity v of a particle at a distance s from a point on its path is given as

s (ft)	0	10	20	30	40	50	60
v (ft/sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft using Simpson's $1/3$ rd Rule.

Sol:- $v = \frac{ds}{dt} \Rightarrow dt = \frac{1}{v} ds$

$$\Rightarrow t = \int_{s=0}^{60} \frac{1}{v} ds$$

$\therefore$ the inverse of given values of v is taken as $y_0, y_1, \dots, y_6$
 $\{y = 1/v\}$

$$h = \frac{b-a}{n} = \frac{60-0}{6} = 10 //$$

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{10}{3} \left[\left(\frac{1}{47} + \frac{1}{38} \right) + 4 \left(\frac{1}{58} + \frac{1}{65} + \frac{1}{52} \right) + 2 \left(\frac{1}{64} + \frac{1}{61} \right) \right]$$

$$= \frac{10}{3} (0.04759 + 0.20742 + 0.06403)$$

$$I = 1.0635 \text{ seconds} //$$

- 10) Find the approximate mileage travelled by a train between 11:50 A.M. and 12:30 P.M. from the given data.

Time	11:50	12:00	12:10	12:20	12:30
Speed (mph)	24.2	35	41.3	42.8	39.2

Sol:- $v = \frac{ds}{dt} \Rightarrow ds = v dt \Rightarrow s = \int_{11:50}^{12:30} v dt$

Mileage = distance covered

$$h = \frac{40}{4} = 10 \text{ minutes } \left(= \frac{1}{6} \text{ hr} \right)$$

By ~~the~~ Simpson's Rule,

$$I = \frac{h}{3} \left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2) \right]$$

$$= \frac{10}{3} \left[(24.2 + 39.2) + 4(35 + 42.8) + 2(41.3) \right]$$

$$= \frac{10}{3} (63.4 + 311.2 + 82.6)$$

$$I = \cancel{25.4 \text{ miles}} \quad 1524 \text{ miles}$$

$$\Rightarrow S = 1524 \text{ miles}$$

$$S = 25.4 \text{ miles}$$

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11)

A river is 80 ft wide. The depth d in ft at a distance x ft from one bank of river is given as :

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find ~~depth~~ approximately the area of cross section of the river.

Sol:-

The area of the cross section of the river $= \int_{x=0}^{80} (y = \text{depth}) dx$

$$h = \frac{80-0}{8} = 10$$

By Simpson's Rule,

$$I = \frac{h}{3} \left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) \right]$$

$$= \frac{10}{3} [80 + 144 + 66]$$

$$= 710 \text{ ft}$$

- 12) A plane area is bounded by a curve, the x -axis and the 2 extreme ordinates. The area is divided into 6 parts by equidistant ordinates 2 inches apart, the heights of the ordinates being given as 21.65, 21.04, 20.35, 19.61, 18.75, 17.80 & 16.75 respectively. Find an approximate value of area using Simpson's rule.

Sol:-

$$n = 6, h = 2$$

$$y = \begin{matrix} 21.65 & 21.04 & 20.35 & 19.61 & 18.75 & 17.80 & 16.75 \\ y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{matrix}$$

$$I = \text{Area} = \int y \, dx$$

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{2}{3} [(21.65 + 16.75) + 4(21.04 + 19.61 + 17.80) + 2(20.35 + 18.75)]$$

$$= \frac{2}{3} (38.4 + 233.8 + 78.2)$$

$$I = 233.6 \text{ inches}^2 //$$

- 13) A solid of revolution is formed by rotating about a x -axis, the area b/w the x -axis, the lines $x=0$ and $x=1$, and a curve through the points with the following coordinates:

x	0	0.25	0.5	0.75	1.0
y	1.00	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid using Simpson's Rule.

Sol:-

$$\text{Volume} = V = \int_0^1 \pi y^2 \, dx$$

$\therefore$ The square of y values are taken in the calculation.

$$\Rightarrow V = \pi \int_0^1 y^2 \, dx = I$$

$$h = \frac{1}{4}$$

$$y = y^2 \quad 1 \quad 0.9793 \quad 0.9194 \quad 0.8260 \quad 0.7081$$

By Simpson's Rule,

$$I = \frac{\pi h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$= \frac{\pi}{12} [1.7081 + 7.2212 + 1.8388]$$

$$= \pi(0.8973)$$

$$I = 2.8190 \text{ cu. units}$$

14) $\int_0^6 \frac{\sin x}{x} dx$ by $n=6$ using Simpson's Rule.

Sol:- $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ { diff. numerator separately & denominator separately }

at $x=0, y=1$

$$h = \frac{6-0}{6} = 1^c$$

x	0	1 ^c	2 ^c	3 ^c	4 ^c	5 ^c	6 ^c
y	1	0.8414	0.4546	0.0470	-0.1812	-0.1917	-0.0465

By Simpson's Rule,

$$I = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3} [(1 - 0.0465) + 4(0.8414 + 0.0470 - 0.1917) + 2(0.4546 - 0.1812)]$$

$$= \frac{1}{3} (0.9535 + 2.7868 + 0.5308)$$

$$I = 1.4237$$

* Numerical Solution of O.D.E

Runge-Kutta Method of 4th order :-

Consider $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$. Let h be the step length

Then, to find $x_1 = x_0 + h$, then to find y_1 , we have the following formulae i.e.

$$y_1 = y_0 + K$$

$$\text{where } K \text{ is } \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = h \times f(x_0, y_0)$$

$$K_2 = h \times f\left(\left(x_0 + \frac{h}{2}\right), \left(y_0 + \frac{K_1}{2}\right)\right)$$

$$K_3 = h \times f\left[\left(x_0 + \frac{h}{2}\right), \left(y_0 + \frac{K_2}{2}\right)\right]$$

$$K_4 = h \times f(x_0 + h, (y_0 + K_3))$$

NOTE: To find y_2 at x_2 i.e. ~~$y_2 = y_1$~~ $x_2 = x_1 + h$, replace (x_0, y_0) with (x_1, y_1) in the above set of formulae.

1) Solve $(x+y)y' = 1$, given $y(0.4) = 1$. Find y at $x = 0.5$

Sol:- $(x+y)y' = 1$; $x_0 = 0.4$
 $\Rightarrow \frac{dy}{dx} = \frac{1}{x+y}$ $y_0 = 1$
 $= f(x, y)$ $h = x - x_0 = 0.5 - 0.4 = 0.1$

$$K_1 = h \times f(x_0, y_0) = 0.1 \times f(0.4, 1) = 0.1 \times \frac{1}{(1+0.4)}$$

$$\Rightarrow K_1 = 0.0714 //$$

$$K_2 = h \times f\left(\left(x_0 + \frac{h}{2}\right), \left(y_0 + \frac{K_1}{2}\right)\right) = 0.1 \times f(0.4 + 0.05, (1 + 0.0357))$$

$$\Rightarrow K_2 = 0.1 \times f(0.45, 1.0357) = 0.1 \times \frac{1}{0.45 + 1.0357}$$

$$\Rightarrow K_2 = 0.0673 //$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= 0.1 f(0.45, 1.0336)$$

$$= 0.1 \times \frac{1}{0.45 + 1.0336}$$

$$0.45 + 1.0336$$

$$K_3 = 0.0674 //$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$

$$= 0.1 f(0.5, 1.0674)$$

$$\Rightarrow K_4 = 0.0637 //$$

$$\therefore K = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6} (0.0714 + 0.1346 + 0.1348 + 0.0637)$$

$$K = 0.0674 //$$

$$y_1 = y_0 + K$$

$$\Rightarrow y_1 = 1 + 0.0674$$

$$y = 1.0674 //$$

2) Solve $y' = x + y^2$, $y(0) = 1$, as $x = 0.2$ in steps of 0.1

Sol:- $\frac{dy}{dx} = x + y^2 = f(x, y)$

$$x_0 = 0 ; y_0 = 1$$

I stage:

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$K_1 = h \times f(x_0, y_0) = 0.1 \times f(0.1) = 0.1 //$$

$$K_2 = h \times f\left(\left(x_0 + \frac{h}{2}\right), \left(y_0 + \frac{K_1}{2}\right)\right) = 0.1 \times f(0.05, 1.05)$$

$$\Rightarrow K_2 = 0.1152 //$$

$$K_3 = h \times f\left(\left(x_0 + \frac{h}{2}\right), \left(y_0 + \frac{K_2}{2}\right)\right) = 0.1 \times f(0.05, 1.0576)$$

$$\Rightarrow K_3 = 0.1168 //$$

$$K_4 = h \times f(x_0 + h, y_0 + K_3) = 0.1 \times f(0.1, 1.1168)$$

$$\Rightarrow K_4 = 0.1347 //$$

$$\therefore K = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$K = 0.11645 //$$

$$y_1 = y_0 + K$$

$$y_1 = 1 + 0.11645$$

$$y_1 = 1.11645 //$$

II stage:

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2 \quad ; \quad x_1 = 0.1, y_1 = 1.11645$$

$$K_1 = h \times f(x_1, y_1) = 0.1 (0.1 + (1.11645)^2) = 0.1346 //$$

$$K_2 = h \times f\left(\left(x_1 + \frac{h}{2}\right), \left(y_1 + \frac{K_1}{2}\right)\right) = 0.1 \times f(0.15, 1.18375)$$

$$\Rightarrow K_2 = 0.1551 //$$

$$K_3 = h \times f\left(\left(x_1 + \frac{h}{2}\right), \left(y_1 + \frac{K_2}{2}\right)\right) = 0.1 \times f(0.15, 1.1940)$$

$$\Rightarrow K_3 = 0.1575 //$$

$$K_4 = h \times f(x_1 + h, y_1 + K_3) = 0.1 \times f(0.2, 1.27395) = 0.1822 //$$

$$K = \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$\Rightarrow K = 0.1570$$

$$y_2 = y_1 + K = 1.11645 + 0.1570$$

$$\Rightarrow y_2 = 1.2736 //$$

NOTE-① The Fourier series converges to $f(x)$ if $f(x)$ is continuous at the point x . If $f(x)$ is discontinuous at a point x in an interval, then the series converges to the Arithmetic mean of the left hand limit & right hand limit of $f(x)$ at the point.

② The formulae given for the constants a_0, a_n, b_n are called Euler's formulae

③ $\cos(n\pi) = (-1)^n$

$\sin(n\pi) = 0$

$\cos \frac{(2n+1)\pi}{2} = 0$

$\sin \frac{(2n+1)\pi}{2} = (-1)^n$

④ $\int e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$

⑤ Integration by parts, ILATE =

$\int (I)(II) dx = I \int II - \int (I) \cdot d(II)$

⑥ The Generalised Rule for integration by parts - ^{Bernoulli's} ~~Leibnitz~~ Rule

$\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots \dots$ until $u^n = 0$
 suffix $\rightarrow$ integration order of v

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⑦ In the interval $[0, 2\pi]$

a) If $f(2\pi - x) = f(x)$ then $f(x)$ is even

If $f(2\pi - x) = -f(x)$ then $f(x)$ is odd

⑧ $\int_0^{2a} f(x) dx = \begin{cases} 2 \int_0^a f(x) dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

1) Obtain the Fourier expansion of $f(x) = e^{-ax}$ in the interval $[0, 2\pi]$

Sol: $f(x) = \frac{a_0}{2} + \sum \frac{a_n \cos\left(\frac{n\pi x}{L}\right)}{L} + \sum \frac{b_n \sin\left(\frac{n\pi x}{L}\right)}{L}$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum \frac{a_n \cos\left(\frac{n\pi x}{\pi}\right)}{\pi} + \sum \frac{b_n \sin\left(\frac{n\pi x}{\pi}\right)}{\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where :

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} e^{-ax} dx = \frac{1}{\pi} \left[\frac{e^{-ax}}{-a} \right]_0^{2\pi} \\ &= \frac{1}{\pi} \left(-\frac{1}{a} \right) \left[e^{-2\pi a} - e^{-a(0)} \right] \\ &= -\frac{1}{a\pi} \left[e^{-2\pi a} - 1 \right] = \frac{1 - e^{-2\pi a}}{a\pi} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx = \frac{1}{\pi} \int_0^{2\pi} e^{-ax} \cos(nx) dx \\ &= \frac{1}{\pi} \frac{e^{-ax}}{a^2 + n^2} \left[(a) \cos nx + n \sin nx \right] \Big|_0^{2\pi} \\ &= \frac{1}{\pi(a^2 + n^2)} \left[e^{-2\pi a} (-a \cos(2n\pi) + n \sin(2n\pi)) - (e^0 (-a \cos 0 + n \sin 0)) \right] \end{aligned}$$

$$= \frac{1}{(a^2 + n^2)\pi} \left[e^{-2\pi a} (-a) - (1(-a)) \right]$$

$$= \frac{1}{(a^2 + n^2)\pi} \frac{a - ae^{-2\pi a}}{1} = \frac{a(1 - e^{-2\pi a})}{(a^2 + n^2)\pi}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} e^{-ax} \sin nx \, dx$$

$$= \frac{1}{\pi} \frac{e^{-ax}}{a^2 + n^2} \left[(-a) \sin nx - n \cos nx \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \frac{1}{a^2 + n^2} \left[e^{-a2\pi} (-a \sin(2n\pi) - n \cos(2n\pi)) - e^0 (-a \sin 0 - n \cos 0) \right]$$

$$= \frac{1}{(a^2 + n^2)\pi} \left[e^{-a2\pi} (-n) - 1(-n) \right]$$

$$b_n = \frac{n}{(a^2 + n^2)\pi} [1 - e^{-2a\pi}]$$

$$\therefore f(x) = \frac{1 - e^{-2a\pi}}{2a\pi} + \sum_{n=1}^{\infty} \frac{a}{\pi(a^2 + n^2)} (1 - e^{-2\pi a}) \cos(nx) + \sum_{n=1}^{\infty} \frac{n}{\pi(a^2 + n^2)} (1 - e^{-2\pi a}) \sin(nx)$$

2) $f(x) = x(2\pi - x)$ in $(0, 2\pi)$.

Sol:- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ $\{l = \pi \therefore 2l = 2\pi\}$

where,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} x(2\pi - x) \, dx = \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) \, dx$$

$$= \frac{1}{\pi} \left[\frac{2\pi x^2}{2} - \frac{x^3}{3} \right]_0^{2\pi} = \frac{1}{\pi} \left[\pi (2\pi)^2 - \frac{(2\pi)^3}{3} \right]$$

$$= \frac{1}{\pi} \left[4\pi^3 - \frac{8\pi^3}{3} \right] = \frac{1}{\pi} \left[\frac{12\pi^3 - 8\pi^3}{3} \right] = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{2\pi} x(2\pi - x) \cos(nx) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \underbrace{(2\pi x - x^2)}_u \underbrace{\cos nx}_v dx$$

Applying Bernoulli's Rule of integration,

$$= \frac{1}{\pi} \left[(2\pi x - x^2) \frac{\sin nx}{n} - (2\pi - 2x) \left(\frac{-\cos nx}{n^2} \right) + (-2) \left(\frac{-\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[(2\pi x - x^2) \frac{\sin nx}{n} + (2\pi - 2x) \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[(2\pi(2\pi) - 4\pi^2) \frac{\sin 2n\pi}{n} + (2\pi - 4\pi) \frac{\cos 2n\pi}{n^2} + 2 \frac{\sin 2n\pi}{n^3} - \left(\cancel{0} + (2\pi) \frac{\cos 0}{n^2} + 2 \frac{\sin 0}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{-2\pi}{n^2} - \frac{2\pi}{n^2} \right] = \frac{1}{\pi} \left[\frac{-2\pi - 2\pi}{n^2} \right]$$

$$a_n = \frac{-4\pi}{n^2\pi} = \frac{-4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{2\pi} x(2\pi - x) \sin(nx) dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) \sin(nx) dx$$

Applying Bernoulli's Rule

$$b_n = \frac{1}{\pi} \left[(2\pi x - x^2) \left(\frac{-\cos nx}{n} \right) - (2\pi - 2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[(2\pi(2\pi) - 4\pi^2) \left(\frac{-\cos 2n\pi}{n} \right) - (2\pi - 4\pi) \left(\frac{-\sin 2n\pi}{n^2} \right) + \right.$$

$$\left. (-2) \left(\frac{\cos 2n\pi}{n^3} \right) \right]$$

$$= \cancel{(-2)} - \left\{ (0) - (2\pi) \left(-\frac{\sin 0}{n^2} \right) + (-2) \frac{\cos 0}{n^3} \right\}$$

$$= \frac{1}{\pi} \left[2\pi(0) - \frac{2}{n^3} + \frac{2}{n^3} \right]$$

$$\underline{b_n = 0}$$

$$\therefore f(x) = \frac{4\pi^2}{3(2)} + \sum_{n=1}^{\infty} \left(-\frac{4}{n^2} \right) \cos nx + 0$$

$$f(x) = \frac{2\pi^2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left(-\frac{4}{n^2} \right) \cos nx$$

(OR)

$$\cancel{\frac{1}{2}} f(2\pi - x) = f(x) \Rightarrow x(2\pi - x) \Rightarrow (2\pi - x)(2\pi - 2\pi + x)$$

$$\Rightarrow (2\pi - x)(x)$$

$$\Rightarrow f(x)$$

$$\Rightarrow \text{Since } f(x) = f(2\pi - x) \Rightarrow f(x) \text{ is even}$$

$$\text{then, } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x(2\pi - x) dx$$

$$= \frac{2}{\pi} \left\{ \frac{2\pi x^2}{2} - \frac{x^3}{3} \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left\{ \pi^3 - \frac{\pi^3}{3} \right\} = \frac{4\pi^3}{3\pi} = \frac{4\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx \Rightarrow \text{integral is even since } \cos(nx) \text{ is even}$$

$$\Rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} x(2\pi - x) \cos nx dx = -\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \Rightarrow \text{integral is odd since } \sin(nx) \text{ is odd}$$

$$\therefore \underline{b_n = 0}$$

3) $f(x) = \frac{\pi - x}{2}$ in $[0, 2\pi]$

Sol 1. $f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{\pi - 2\pi + x}{2} = -\frac{(\pi - x)}{2} = -f(x)$

$\Rightarrow f(x)$ is odd-like

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx \Rightarrow \text{the integral is odd since } \cos(nx) \text{ is even}$$

$$\therefore a_n = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx \Rightarrow \text{the integral is even since } \sin(nx) \text{ is odd}$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \frac{\pi - x}{2} \sin(nx) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin(nx) dx$$

By Bernoulli's Rule,

$$b_n = \frac{1}{\pi} \left[(\pi - x) \left(\frac{-\cos nx}{n} \right) - (-1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[(\pi - \pi) \left(\frac{-\cos n\pi}{n} \right) - \frac{\sin n\pi}{n^2} - \left\{ (\pi) \left(\frac{-\cos 0}{n} \right) - \frac{\sin 0}{n^2} \right\} \right]$$

$$= \frac{1}{\pi} \left[+\pi \right] = \frac{1}{n}$$

$$\therefore f(x) = 0 + 0 + \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

4) $f(x) = x$
 5) $f(x) = x^3$ } in $[0, 2\pi]$

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NOTE:

In the interval $[-\pi, \pi]$

$f(x)$ is even if $f(-x) = f(x)$

$f(x)$ is odd if $f(-x) = -f(x)$

(a) $\int_{-\pi}^{\pi} f(x) dx = \begin{cases} 2 \int_0^{\pi} f(x) dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

(b) If $f(x)$ is even, $b_n = 0$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$

(c) If $f(x)$ is odd, $a_0 = 0$, $a_n = 0$

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

(d) If $f(x)$ is neither even nor odd then Fourier series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$

1) Find the Fourier series of $f(x) = x - x^2$ in $[-\pi, \pi]$.

Sol:- $f(-x) = (-x) - (-x)^2$
 $= -x - x^2 \neq f(x)$
 $= -(x + x^2) \neq -f(x)$

$\therefore f(x)$ is neither odd nor even.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{1}{2} (\pi^2 - \pi^2) - \frac{1}{3} (\pi^3 + \pi^3) \right] \\ &= \frac{1}{\pi} \left[-\frac{2\pi^3}{3} \right] = \underline{\underline{\frac{-2\pi^2}{3}}} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx \\ &= \frac{1}{\pi} \left[(x - x^2) \frac{\sin nx}{n} - (1 - 2x) \frac{(-\cos nx)}{n^2} + (-2) \frac{(-\sin nx)}{n^3} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{1}{n^2} (1 - 2x) \cos nx \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{1}{n^2} (1 - 2\pi) \cos n\pi - (1 - 2(-\pi)) \cos(-n\pi) \right] \\ &= \frac{1}{\pi n^2} \left[(1 - 2\pi)(1) - (1 + 2\pi)(1) \right] \cos n\pi \\ &= \frac{\cos n\pi (-4\pi)}{\pi n^2} = \frac{(-1)^n (-4\pi)}{\pi n^2} = \underline{\underline{\frac{(-1)^n (-4)}{n^2}}} \end{aligned}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \left[(x-x^2) \left(\frac{-\cos nx}{n} \right) - (1-2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi} \\
 &= \frac{1}{\pi} \left[\frac{-1}{n} \left((\pi-\pi^2) \cos n\pi - (-\pi - (-\pi)^2) \cos(-n\pi) \right) - \frac{2}{n^3} \right. \\
 &\quad \left. (\cos n\pi - \cos(-n\pi)) \right] \\
 &= \frac{1}{\pi} \left[\frac{-1}{n} \cos n\pi (\pi - \pi^2 - (-\pi - \pi^2)) \right] \\
 &= \frac{1}{\pi} \left[\frac{-1}{n} \cos n\pi (\pi - \pi^2 + \pi + \pi^2) \right] \\
 &= \frac{(-1)(-1)^n (2\pi)}{\pi n} = \frac{2(-1)^{n+1}}{n}
 \end{aligned}$$

$$\therefore f(x) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n (-4)}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

is the Fourier series.

2) Find the Fourier series of $f(x) = x^2$ in $[-\pi, \pi]$

Sol:- $f(-x) = (-x)^2 = x^2 = f(x)$
 $\therefore f(x)$ is even

In the Fourier series, $b_n = 0$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi} \left[\frac{\pi^3}{3} \right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n} \right) - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

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4) $f(x)$
Sol:-

$$= \frac{2}{\pi} \left[2 \left(\frac{\cos n\pi}{n^2} \right) - \frac{\cos 0}{n^2} \right]$$

$$= \frac{2}{\pi n^2} [2\pi (-1)^n - 0]$$

$$= \frac{4\pi}{\pi n^2} (-1)^n$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx //$$

$$3) f(x) = \pi^2 - x^2$$

Sol:-

$$f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$$

 $\therefore f(x)$ is evenIn the Fourier series, $b_n = 0$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\text{where, } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi^2 - x^2) dx$$

$$= \frac{2}{\pi} \left[\pi^2 x - \frac{x^3}{3} \right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi} (-\pi^3) = -2\pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} (\pi^2 - x^2) \cos nx dx$$

$$= \frac{2}{\pi} \left[(\pi^2 - x^2) \left(\frac{\sin nx}{n} \right) - (0 - 2x) \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \int_0^{\pi} 2x$$

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5)

Sol:-

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4) $f(x) = |x|$ in $(-\pi, \pi)$

Sol: $|x| = \begin{cases} x & \text{if } 0 < x < \pi \\ -x & \text{if } -\pi < x < 0 \end{cases}$

$$|x| = \begin{cases} x & , x > 0 \\ -x & , x < 0 \end{cases}$$

$|x|$ is even function

$\Rightarrow$ In Fourier series, $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$

$$a_0 = \frac{2}{\pi} \cdot \frac{(\pi^2)}{2} = \underline{\underline{\pi}}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi} \left[x \left(\frac{\sin nx}{n} \right) - 1 \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\cos nx}{n^2} \right]_0^{\pi} = \frac{2}{n^2 \pi} (\cos \pi n - \cos 0)$$

$$a_n = \frac{2}{\pi n^2} ((-1)^n - 1) = \frac{2(-1)^{n-1}}{\pi n^2}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n-1} \cos nx}{\pi n^2}$$

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5) Obtain the Fourier series of $f(x) = x \sin x$ in the range $(-\pi, \pi)$

Sol: $f(-x) = (-x) \sin(-x)$

$$f(-x) = x \sin x = f(x)$$

then $f(x)$ is even function

then $b_n = 0$ in Fourier series

$\therefore$ Fourier series is, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$ & $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$

$$\Rightarrow a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x dx = \frac{2}{\pi} \left[x(-\cos x) - 1(-\sin x) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-(\pi \cos \pi - 0) + (0) \right]$$

$$= -2(-1)$$

$$a_0 = 2$$

$$\Rightarrow a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x (\cos nx \cdot \sin x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{x}{2} (\sin(n+1)x - \sin(n-1)x) \right) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin((n+1)x) dx - \frac{1}{\pi} \int_0^{\pi} x \sin((n-1)x) dx$$

$$= \frac{1}{\pi} \left[x \left(\frac{-\cos((n+1)x)}{n+1} \right) - 1 \cdot \left(\frac{-\sin((n+1)x)}{(n+1)^2} \right) \right]_0^{\pi}$$

$$- \frac{1}{\pi} \left[x \left(\frac{-\cos((n-1)x)}{n-1} \right) - 1 \cdot \left(\frac{-\sin((n-1)x)}{(n-1)^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{-1}{n+1} [\pi \cos((n+1)\pi) - 0] \right] + \frac{1}{\pi} \left[\frac{-1}{n-1} [\pi \cos((n-1)\pi) - 0] \right]$$

$$= \frac{1}{\pi} \pi \left[\frac{-(-1)^{n+1}}{n+1} + \frac{-(-1)^{n-1}}{n-1} \right]$$

$$= \left[\frac{-(-1)^{n-1}}{n-1} - \frac{-(-1)^{n+1}}{n+1} \right] = (-1)^n \left[\frac{-1}{n-1} + \frac{1}{n+1} \right]$$

$$= (-1)^n \frac{-n-1+n-1}{(n-1)(n+1)} = (-1)^n \frac{(-2)}{n^2-1} \quad \text{for } n \neq 1$$

~~for n~~ $\therefore$ we find a_1

$\Rightarrow$ Put $n=1$ in an

$$a_1 = \frac{2}{\pi} \int_0^{\pi} f(x) \cos x \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi} \left[\frac{x (-\cos 2x)}{2} - \frac{1 (-\sin 2x)}{4} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{-1}{2} (\pi (\cos 2\pi) - 0) + \frac{1}{4} (\sin 2\pi - 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{-1}{2} (\pi) \right]$$

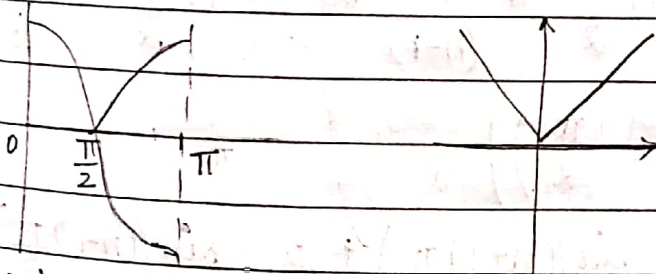
$$a_1 = -\frac{1}{2}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$f(x) = 1 + \left(-\frac{1}{2}\right) \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

6) $f(x) = |\cos x|$ in $(-\pi, \pi)$

Sol:- $f(x) = |\cos(-x)| = |\cos x| = f(x) \Rightarrow$ even function
 $\Rightarrow b_n = 0$ in Fourier series



NOTE:-

$$|\cos x| = \begin{cases} \cos x, & 0 \leq x \leq \pi/2 \\ -\cos x, & \pi/2 \leq x \leq \pi \end{cases}$$

$$\begin{aligned}
 \therefore a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} |\cos x| dx \\
 &= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x dx + \int_{\pi/2}^{\pi} -\cos x dx \right] \\
 &= \frac{2}{\pi} \left[[\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} \right] \\
 &= \frac{2}{\pi} \left[\left(\sin \frac{\pi}{2} - \sin 0 \right) - \left(\sin \pi - \sin \frac{\pi}{2} \right) \right] \\
 &= \frac{2}{\pi} [1 + 1] = \frac{4}{\pi} //
 \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos nx dx + \int_{\pi/2}^{\pi} (-\cos x) (\cos nx) dx \right] \quad \text{using } \cos A \cos B$$

$$\begin{aligned}
 &= \frac{2}{\pi} \left[\int_0^{\pi/2} \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx - \frac{2}{\pi} \left[\int_{\pi/2}^{\pi} \frac{1}{2} [\cos(n+1)x + \cos(n-1)x] dx \right] \right] \\
 &= \frac{2}{\pi} \left(\frac{1}{2} \right) \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\pi/2}^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1} \right]_{\pi/2}^{\pi}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[\frac{1}{(n+1)} \frac{\sin(n+1)\frac{\pi}{2}}{2} + \frac{1}{(n-1)} \frac{\sin(n-1)\frac{\pi}{2}}{2} - \left(\frac{1}{(n+1)} (0 - \sin(n+1)\frac{\pi}{2}) \right. \right. \\
 &\quad \left. \left. - \frac{1}{n-1} (0 - \sin(n-1)\frac{\pi}{2}) \right) \right]
 \end{aligned}$$

$$= \frac{1}{\pi} \left[\frac{2 \sin((n+1)\frac{\pi}{2})}{n+1} + \frac{2 \sin((n-1)\frac{\pi}{2})}{n-1} \right]$$

$$\cos 2x = 1 - 2\cos^2 x$$

$$\cos 2x - 1 = -2\cos^2 x$$

$$\frac{1 - \cos 2x}{2} = \cos^2 x$$

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NOTE:

$$\Rightarrow \sin((n-1)\pi/2) = \sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) = \sin\left(-\left(\frac{\pi}{2} - \frac{n\pi}{2}\right)\right)$$

$$= -\sin\left(\frac{\pi}{2} - \frac{n\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{1}{\pi} \left[\frac{2}{n+1} \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n-1} \cos\left(\frac{n\pi}{2}\right) \right]$$

$$a_n = \frac{2}{\pi} \cos\left(\frac{n\pi}{2}\right) \left[\frac{n-1 - (n+1)}{n^2 - 1} \right]$$

$$a_n = \frac{2}{\pi} \cos\left(\frac{n\pi}{2}\right) \left(\frac{-2}{n^2 - 1} \right) \text{ for } n \neq 1$$

To find $a_1 \Rightarrow$ put $n=1$ in a_n

$$a_1 = \frac{2}{\pi} \int_0^\pi f(x) \cos x dx = \frac{2}{\pi} \left[\int_0^{\pi/2} \cos x \cos x dx + \int_{\pi/2}^\pi (-\cos x) (\cos x) dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^\pi \cos^2 x dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \frac{1 + \cos 2x}{2} dx - \int_{\pi/2}^\pi \frac{1 + \cos 2x}{2} dx \right]$$

$$= \frac{2}{\pi} \left(\frac{1}{2} \right) \left[\left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} - \left[x + \frac{\sin 2x}{2} \right]_{\pi/2}^\pi \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{\pi}{2} + \frac{\sin \pi}{2} - 0 \right] - \left[\frac{\pi}{2} + \frac{\sin 2\pi}{2} - \frac{\pi}{2} - \frac{\sin \pi}{2} \right] \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} (0 - 0) \right]$$

$$a_1 = 0$$

$$\therefore f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = \frac{2}{\pi} + 0 + \sum_{n=1}^{\infty} \frac{2}{\pi} \cos\left(\frac{n\pi}{2}\right) \left(\frac{-2}{n^2 - 1} \right) \cos nx //$$

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7) $\sin mx$, where m is not an integer in $[-\pi, \pi]$

Sol:- $f(-x) = \sin(-mx) = -\sin mx$
 $= -f(x)$

 $\therefore f(x)$ is odd $\Rightarrow a_n = 0$ and $a_0 = 0$ in Fourier series

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \sin mx \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \frac{1}{2} [\cos(m-n)x - \cos(m+n)x] \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \cos(m-n)x \, dx - \int_0^{\pi} \cos(m+n)x \, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{\sin(m-n)x}{m-n} \right]_0^{\pi} - \left[\frac{\sin(m+n)x}{m+n} \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{m-n} \sin(m-n)\pi - \frac{1}{m+n} \sin(m+n)\pi \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{m-n} (\sin m\pi \cos n\pi - \cos m\pi \sin n\pi) - \frac{1}{m+n} (\sin m\pi \cos n\pi + \cos m\pi \sin n\pi) \right]$$

$$\left. \begin{aligned} &\left[\sin n\pi = 0 \text{ but } \sin m\pi \neq 0 \right] \\ &\therefore m \text{ is not an integer} \end{aligned} \right\}$$

$$= \frac{1}{\pi} [\sin m\pi \cos n\pi] \left(\frac{1}{m-n} - \frac{1}{m+n} \right)$$

$$= \frac{\sin m\pi (-1)^n}{\pi} \left(\frac{m+n - m+n}{m^2 - n^2} \right)$$

$$b_n = \frac{(-1)^n \sin m\pi}{\pi} \left(\frac{2n}{m^2 - n^2} \right)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \sin m\pi}{\pi} \left(\frac{2n}{m^2 - n^2} \right) \sin nx$$

6) $\cos ax$, where a is not an integer in $[-\pi, \pi]$

Sol:-

$$f(-x) = \cos(-ax) = \cos ax = f(x)$$

$\therefore f(x)$ is even

$\Rightarrow b_n = 0$ in Fourier series

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned} \text{where } a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \cos ax dx \\ &= \frac{2}{\pi} \left[\frac{\sin ax}{a} \right]_0^{\pi} = \frac{2}{\pi a} [\sin a\pi - \sin 0] \end{aligned}$$

$$a_0 = \frac{2}{\pi a} (\sin a\pi)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \cos ax \cos nx dx = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\cos(n+a)x + \cos(n-a)x] dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \cos(n+a)x dx + \int_0^{\pi} \cos(n-a)x dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{\sin(n+a)x}{(n+a)} \right]_0^{\pi} + \left[\frac{\sin(n-a)x}{(n-a)} \right]_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n+a} (\sin(n+a)\pi - \sin 0) + \frac{1}{n-a} (\sin(n-a)\pi - \sin 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n+a} (\cancel{\sin n\pi} \cos a\pi + \cos n\pi \sin a\pi) + \frac{1}{n-a} (\cancel{\sin n\pi} \cos a\pi - \cos n\pi \sin a\pi) \right]$$

$$= \frac{1}{\pi} \sin a\pi (-1)^n \left(\frac{1}{n+a} - \frac{1}{n-a} \right) = \frac{(-1)^n \sin a\pi}{\pi} \left[\frac{n-a - n-a}{n^2 - a^2} \right]$$

$$a_n = \frac{(-1)^n \sin a\pi}{\pi} \frac{(-2a)}{n^2 - a^2}$$

$$\therefore f(x) = \frac{1}{2\pi} (\sin a\pi) + \sum_{n=1}^{\infty} \frac{(-1)^n \sin a\pi (-2a)}{\pi (n^2 - a^2)} \cos nx //$$

* PROBLEMS ON DISCONTINUITY

NOTE:-

$$f(x) = \begin{cases} \phi(x), & -\pi < x < 0 \\ \psi(x), & 0 < x < \pi \end{cases}$$

In $[-\pi, \pi]$ if $\phi(-x) = \psi(x) \Rightarrow f(x)$ is even
 $\phi(-x) = -\psi(x) \Rightarrow f(x)$ is odd

1) $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi}, & 0 \leq x \leq \pi \end{cases}$

sol:- $\phi(x) = 1 + \frac{2x}{\pi} \Rightarrow \phi(-x) = 1 - \frac{2x}{\pi} = \psi(x)$

$\therefore f(x)$ is even

$\Rightarrow b_n = 0$ in the Fourier series

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} \left[\left(x - \frac{2}{\pi} \frac{x^2}{2}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\pi - \frac{\pi^2}{\pi} \right] = \frac{2}{\pi} [\pi - \pi] = 0$$

$$\Rightarrow a_0 = 0$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos nx - \frac{2x}{\pi} \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \cos nx dx - \int_0^{\pi} \frac{2x}{\pi} \cos nx dx \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{\sin n\pi}{n} \right)_0^\pi - \left[\frac{2x}{\pi} \left(\frac{\sin n\pi}{n} \right) - \frac{2}{\pi} \left(\frac{-\cos n\pi}{n^2} \right) \right]_0^\pi \right]$$

$$= \frac{2}{\pi} \left[\left(\frac{\sin n\pi}{n} - 0 \right) - \left(\frac{2}{n\pi} (\sin n\pi - 0) + \frac{2}{\pi n^2} (-\cos n\pi + \cos 0) \right) \right]$$

$$= \frac{2}{\pi} \left[\frac{2}{\pi n^2} (1 - (-1)^n) \right]$$

$$a_n = \frac{4}{\pi^2} \frac{1 - (-1)^n}{n^2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2} \frac{1 - (-1)^n}{n^2} \cos nx //$$

$$2) f(x) = \begin{cases} x - \frac{\pi}{2} & -\pi < x < 0 \\ x + \frac{\pi}{2} & 0 < x < \pi \end{cases}$$

sol: $\phi(x) = x - \frac{\pi}{2} \Rightarrow \phi(-x) = -x - \frac{\pi}{2} = -\left(x + \frac{\pi}{2}\right)$

$$\phi(x) = -\phi(-x)$$

$\therefore f(x)$ is odd

$\Rightarrow a_0 = 0$ and $a_n = 0$ in Fourier series

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi \left(x + \frac{\pi}{2}\right) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^\pi \left(x \sin nx + \frac{\pi}{2} \sin nx\right) dx$$

$$= \frac{2}{\pi} \left[\int_0^\pi x \sin nx \, dx + \int_0^\pi \frac{\pi}{2} \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\left[\frac{x(-\cos nx)}{n} - 1 \left(\frac{-\sin nx}{n^2} \right) \right]_0^\pi + \frac{\pi}{2} \left(\frac{-\cos nx}{n} \right)_0^\pi \right]$$

$$= \frac{2}{\pi} \left[\frac{-\pi \cos n\pi}{n} - 0 + \frac{\sin n\pi}{n^2} - 0 + \frac{\pi}{2n} (\cos n\pi - \cos 0) \right]$$

$$= \frac{2}{\pi} \left[\frac{-\pi}{n} (-1)^n - \frac{\pi}{2n} ((-1)^n - 1) \right]$$

$$= \frac{1}{n} - \frac{3(-1)^n}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{1-3(-1)^n}{n} \sin nx$$

$$3) f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Sol:- The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where, } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-\pi) dx + \int_0^{\pi} x dx \right]$$

$$= \frac{1}{\pi} \left[-\pi (x) \Big|_{-\pi}^0 + \frac{(x^2)}{2} \Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[-\pi (0 + \pi) + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} \left[\frac{-\pi^2 + \pi^2}{2} \right] = -\frac{\pi^2}{2} \left(\frac{1}{\pi} \right) = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-\pi) \cos nx dx + \int_0^{\pi} x \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[-\pi \left(\frac{\sin nx}{n} \right) \Big|_{-\pi}^0 + \left[x \frac{\sin nx}{n} - \frac{(-\cos nx)}{n^2} \right] \Big|_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{-\pi}{n} (\sin 0 + \sin n\pi) + \left[\frac{\pi}{n} \sin n\pi - 0 + \frac{\cos n\pi}{n^2} - \cos 0 \right] \right]$$

$$= \frac{1}{\pi} \left[\frac{-\pi}{n} (0) + \frac{\cos n\pi - 1}{n^2} \right]$$

$$a_n = \frac{1}{\pi} \left[\frac{\cos n\pi - 1}{n^2} \right] = \frac{1}{\pi n^2} (-1)^n - 1$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-\pi) \sin nx \, dx + \int_0^{\pi} x \sin nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{-\pi}{n} (-\cos nx) \right]_0^0 + \left[x \left(\frac{\cos nx}{n} \right) - \left(\frac{-\sin nx}{n^2} \right) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{n} \cos(0) - \cos(-\pi n) \right] + \frac{\pi \cos \pi n}{n} - 0 +$$

$$\frac{\sin n\pi}{n^2} - \sin 0$$

$$= \frac{1}{\pi} \left[\frac{\pi}{n} (1 - \cos n\pi) - \frac{\pi \cos n\pi}{n} \right]$$

$$= \frac{1}{\pi} \left(\frac{\pi}{n} \right) [1 - \cos n\pi - \cos n\pi]$$

$$b_n = \frac{1 - 2\cos n\pi}{n} = \frac{1 - 2(-1)^n}{n}$$

$$\therefore f(x) = -\pi + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{1 - 2(-1)^n}{n} \sin nx$$

$$4) f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \left[\int_0^{\pi} \sin x \, dx \right]$$

$$= \frac{1}{\pi} [-\cos x]_0^{\pi} = \frac{1}{\pi} [\cos \pi - \cos 0]$$

$$= \frac{-1}{\pi} [\cos \pi - \cos 0] = \frac{-1}{\pi} (-1 - 1) = \frac{-2}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[\int_0^{\pi} \sin x \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos nx \sin x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(n+1)x - \sin(n-1)x] \, dx$$

$$= \frac{1}{2\pi} \left[\int_0^{\pi} \sin(n+1)x \, dx - \int_0^{\pi} \sin(n-1)x \, dx \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{-\cos(n+1)x}{n+1} \right]_0^{\pi} + \left[\frac{\cos(n-1)x}{(n-1)} \right]_0^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\frac{-1}{n+1} (\cos(n+1)\pi - 1) + \frac{1}{n-1} (\cos(n-1)\pi - 1) \right]$$

$$= \frac{1}{2\pi} \left[\frac{-1}{n+1} ((-1)^{n+1} - 1) + \frac{1}{n-1} ((-1)^{n-1} - 1) \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{+1}{n+1} - \frac{1}{n-1} \right] + \left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} \right] \right]$$

$$= \frac{1}{2\pi} \left[\frac{n-1 - n-1}{n^2-1} + (-1)^n \left(\frac{1}{n+1} - \frac{1}{n-1} \right) \right]$$

$$a_n = \frac{1}{2\pi} \left[\frac{-2}{n^2-1} + \frac{(-1)^n (-2)}{n^2-1} \right] \text{ for } n \neq 1$$

$$a_n = \frac{-1}{\pi} \left[\frac{1 + (-1)^n}{n^2-1} \right] \text{ for } n \neq 1$$

$$\therefore a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx = \frac{1}{\pi} \left[\int_0^{\pi} \sin x \cos x \, dx \right]$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{2\pi} \left[\frac{-\cos 2x}{2} \right]_0^{\pi} = \frac{-1}{4\pi} (1-1)$$

$$a_1 = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} \sin x \sin nx dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \frac{1}{2} (\cos(n-1)x - \cos(n+1)x) dx \right]$$

$$= \frac{1}{2\pi} \int_0^{\pi} (\cos(n-1)x - \cos(n+1)x) dx$$

$$= \frac{1}{2\pi} \left[\left(\frac{\sin(n-1)x}{n-1} \right)_0^{\pi} - \left(\frac{\sin(n+1)x}{n+1} \right)_0^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\frac{\sin(n-1)\pi}{n-1} - 0 - \frac{\sin(n+1)\pi}{n+1} + 0 \right]$$

$$b_n = 0 \text{ for } n \neq 1$$

$$\therefore b_1 = \frac{1}{\pi} \int_0^{\pi} \sin x \sin x dx = \frac{1}{\pi} \int_0^{\pi} \sin^2 x dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$= \frac{1}{2\pi} \left[(x)_0^{\pi} - \left(\frac{\sin 2x}{2} \right)_0^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\pi - \frac{\sin 2\pi}{2} + 0 \right]$$

$$b_1 = \frac{1}{2\pi} [\pi] = \underline{\underline{\frac{1}{2}}}$$

$$\therefore f(x) = \frac{1}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{-1}{\pi} \left[\frac{1+(-1)^n}{n^2-1} \right] \cos nx + \frac{\sin x}{2} + \sum_{n=2}^{\infty} 0$$

$$= \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{-1}{\pi} \left[\frac{1+(-1)^n}{n^2-1} \right] \cos nx + \frac{1}{2} \sin x$$

$$5) f(x) = \begin{cases} -\cos x & -\pi < x < 0 \\ \cos x & 0 < x < \pi \end{cases}$$

sol: $f(-x) = -\cos(-x) = -\cos x$
 $= -f(x)$

$\therefore f(x)$ is odd

$\Rightarrow a_0$ and a_n are 0 in fourier series

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin nx \cos x \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} \sin(n+1)x \, dx + \int_0^{\pi} \sin(n-1)x \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} \sin(n+1)x \, dx + \int_0^{\pi} \sin(n-1)x \, dx \right]$$

$$= \frac{1}{\pi} \left[\left(-\frac{\cos(n+1)x}{n+1} \right)_0^{\pi} - \left(\frac{\cos(n-1)x}{(n-1)} \right)_0^{\pi} \right]$$

$$= -\frac{1}{\pi} \left[\frac{-1}{n+1} (\cos(n+1)\pi - \cos 0) - \frac{1}{n-1} (\cos(n-1)\pi - \cos 0) \right]$$

$$= \frac{1}{\pi} \left[\frac{-1}{n+1} ((-1)^{n+1} - 1) - \frac{1}{n-1} ((-1)^{n-1} - 1) \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{n+1} + \frac{1}{n-1} + (-1)^n \frac{1}{n+1} + (-1)^n \frac{1}{n-1} \right]$$

$$= \frac{1}{\pi} \left[\frac{n-1+n+1}{n^2-1} + (-1)^n \left(\frac{n-1+n+1}{n^2-1} \right) \right]$$

$$b_n = \frac{1}{\pi} \left[\frac{2n}{n^2-1} + (-1)^n \frac{(2n)}{n^2-1} \right] = \frac{2n(1+(-1)^n)}{\pi(n^2-1)} \quad \text{for } n \neq 1$$

$$\therefore b_1 = \frac{2}{\pi} \int_0^{\pi} \cos x \sin x dx = \frac{2}{\pi} \int_0^{\pi} \frac{\sin 2x}{2} dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} [-\cos 2\pi + \cos 0] = \frac{1}{2\pi} [1-1] = \underline{\underline{0}}$$

$$\therefore f(x) = \sum_{n=2}^{\infty} \frac{2n(1+(-1)^n)}{\pi(n^2-1)} \sin nx //$$

$$6) f(x) = \begin{cases} x & 0 \leq x \leq \pi \\ 2\pi - x & \pi \leq x \leq 2\pi \end{cases}$$

$$\text{Sol: } a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \left[\int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x) dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{x^2}{2} \right)_0^{\pi} + 2\pi(x)_{\pi}^{2\pi} - \left(\frac{x^2}{2} \right)_{\pi}^{2\pi} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 2\pi(2\pi - \pi) - \frac{4\pi^2}{2} + \frac{\pi^2}{2} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} + 2\pi^2 - \frac{3\pi^2}{2} \right] = \frac{1}{\pi} \left[\frac{-2\pi^2 + 2\pi^2}{2} \right]$$

$$= \frac{1}{\pi} [\pi^2] = \underline{\underline{\pi}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx = \frac{1}{\pi} \left[\int_0^{\pi} x \cos nx dx + \int_{\pi}^{2\pi} (2\pi - x) \cos nx dx \right]$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi} + \left[(2\pi - x) \frac{\sin nx}{n} - \frac{\cos nx}{n^2} \right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{n} (\sin n\pi - 0) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2} \right) + \frac{(2\pi - 2\pi)}{n} \sin 2n\pi \right.$$

$$\left. - \frac{(2\pi - \pi)}{n} \sin n\pi - \frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{(-1)^{2n}}{n^2} + \frac{(-1)^n}{n^2} \right] = \frac{1}{\pi} \left[\frac{2(-1)^n - (-1)^{2n} - 1}{n^2} \right]$$

$$(-1)^n)^2 = (-1)^{2n}$$

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$$a_n = \frac{(-1)^n - 1}{\pi n^2} \cdot 2$$

$$f(x) = \begin{cases} \pi - x & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$$

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* length of interval is $[-l, l]$ or $[0, 2l]$ $\rightarrow$ Arbitrary period

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

In $[0, 2l]$:

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

In $[-l, l]$:

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

1) Find the fourier series of e^{-x} in $(-l, l)$

Sol:- In $(-l, l)$: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where $a_0 = \frac{1}{l} \int_{-l}^l e^{-x} dx$

$$= \frac{1}{l} \left[\frac{e^{-x}}{-1} \right]_{-l}^l = -\frac{1}{l} [e^{-l} - e^{-(-l)}] \quad \frac{e^x - e^{-x}}{2} = \sinh x$$

$$= -\frac{1}{l} [e^{-l} - e^l] = \frac{e^l - e^{-l}}{l} = \underline{\underline{\frac{2 \sinh l}{l}}}$$

$$a_n = \frac{1}{l} \int_{-l}^l e^{-x} \cos\left(\frac{n\pi x}{l}\right) dx$$

$$a_n = \frac{e^{-x}}{l(-1)^2 + \left(\frac{n\pi}{l}\right)^2} \left[(-1) \cos\left(\frac{n\pi}{l}x\right) + \left(\frac{n\pi}{l}\right) \sin\left(\frac{n\pi}{l}x\right) \right]_{-l}^l$$

$$a_n = \frac{1}{l} \frac{l^2}{l^2 + n^2\pi^2} \left[e^{-l} (-\cos\left(\frac{n\pi}{l}l\right)) - e^l (-\cos\left(\frac{n\pi}{l}(-l)\right)) \right]$$

$$= \frac{1}{l^2 + n^2\pi^2} [e^{-l} (-\cos n\pi) + e^l \cos n\pi]$$

$$a_n = \frac{l \cos n\pi}{l^2 + n^2\pi^2} [e^{-l} + e^l]$$

$$b_n = \frac{1}{l} \int_{-l}^l e^{-x} \sin\left(\frac{n\pi}{l}x\right) dx$$

$$= \frac{1}{l} \frac{e^{-x}}{(-1)^2 + \left(\frac{n\pi}{l}\right)^2} \left[(-1) \sin\left(\frac{n\pi}{l}x\right) - \left(\frac{n\pi}{l}\right) \cos\left(\frac{n\pi}{l}x\right) \right]_{-l}^l$$

$$= \frac{1}{l} \frac{l^2}{l^2 + n^2\pi^2} \left[e^{-x} \left(-\sin\left(\frac{n\pi}{l}x\right) \right) - e^{\frac{x}{l}} \left(\frac{n\pi}{l} \cos\left(\frac{n\pi}{l}x\right) \right) \right]_{-l}^l$$

$$= \frac{1}{l^2 + n^2\pi^2} \left[-e^{-l} \left(\frac{n\pi}{l} \right) \cos\left(\frac{n\pi}{l}l\right) + e^l \left(\frac{n\pi}{l} \right) \cos\left(\frac{n\pi}{l}(-l)\right) \right]$$

$$= \frac{1}{l^2 + n^2\pi^2} \left[-\left(\frac{n\pi}{l}\right) (e^{-l} \cos n\pi - e^l \cos n\pi) \right]$$

$$b_n = \frac{n\pi}{l^2 + n^2\pi^2} \cos n\pi (e^l - e^{-l}) = \frac{n\pi}{l^2 + n^2\pi^2} \cos n\pi (2 \sinh l)$$

$$\therefore f(x) = \frac{\sinh l}{l} + \sum_{n=1}^{\infty} \frac{l \cos n\pi}{l^2 + n^2\pi^2} (e^{-l} + e^l) \cos\left(\frac{n\pi}{l}x\right) +$$

$$\sum_{n=1}^{\infty} \frac{n\pi}{l^2 + n^2\pi^2} \cos n\pi (2 \sinh l) \sin\left(\frac{n\pi}{l}x\right)$$

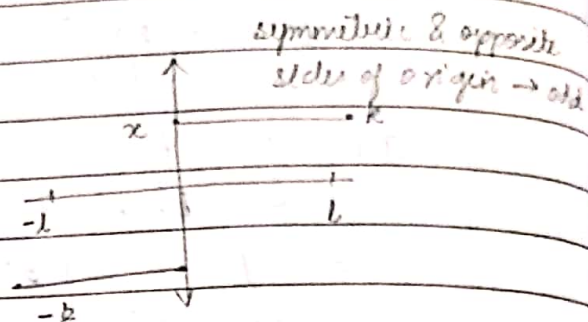
$$2) f(x) = \begin{cases} -k & \text{if } -l < x < 0 \\ k & \text{if } 0 < x < +l \end{cases}$$

Sol:- $f(x)$ is an odd function

$$\therefore a_0 = 0$$

$$a_n = 0$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$



where,

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l k \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2k}{l} \left[-\cos\left(\frac{n\pi x}{l}\right) \right]_0^l = \frac{2k}{l} \left(\frac{l}{n\pi} \right) \left[-\cos\left(\frac{n\pi x}{l}\right) \right]_0^l$$

$$= \frac{2k}{n\pi} \left[-\cos\left(\frac{n\pi}{l} \cdot l\right) + \cos\left(\frac{n\pi}{l} \cdot 0\right) \right]$$

$$b_n = \frac{2k}{n\pi} [1 - \cos(n\pi)]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2k}{n\pi} (1 - \cos(n\pi)) \sin\left(\frac{n\pi x}{l}\right)$$

$$3) f(x) = x^2 - x \text{ in } (-2, 2)$$

Sol:- $f(-x) = (-x)^2 - (-x) = x^2 + x \neq f(x)$
 $= -(x^2 - x) \neq -f(x)$

$\therefore f(x)$ is neither even nor odd

$$\begin{aligned} &\text{if } (0, 3) \\ &2l = 3 \\ &l = 3/2 \end{aligned}$$

$$(-l, l) \Rightarrow (-2, 2) \Rightarrow 2l = 2 - (-2) = 4 \Rightarrow \boxed{l=2}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

where

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx = \frac{1}{2} \int_{-2}^2 (x^2 - x) dx = \frac{1}{2} \left[\left(\frac{x^3}{3} \right)_{-2}^2 - \left(\frac{x^2}{2} \right)_{-2}^2 \right]$$

$$= \frac{1}{2} \left[\frac{8}{3} + \frac{8}{3} - \left(\frac{4}{2} - \frac{4}{2} \right) \right] = \frac{1}{2} \left(\frac{16}{3} \right)$$

$$a_0 = \frac{8}{3}$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \left(\frac{n\pi}{l} \right) x dx = \frac{1}{2} \int_{-2}^2 (x^2 - x) \cos \left(\frac{n\pi}{2} \right) x dx$$

$$= \frac{1}{2} \left[\frac{(x^2 - x) \sin \left(\frac{n\pi}{2} \right) x}{\left(\frac{n\pi}{2} \right)} - (2x - 1) \frac{\left(-\cos \left(\frac{n\pi}{2} \right) x \right)}{\left(\frac{n\pi}{2} \right)^2} \right]_{-2}^2$$

$$+ 2 \left[\frac{\left(-\sin \left(\frac{n\pi}{2} \right) x \right)}{\left(\frac{n\pi}{2} \right)^3} \right]_{-2}^2$$

$$= \frac{1}{2} \left[(2x - 1) \cos \left(\frac{n\pi}{2} \right) x \left(\frac{1}{\frac{n^2 \pi^2}{4}} \right) \right]_{-2}^2$$

$$= \frac{1}{2} \left[\frac{4(2x - 1)}{n^2 \pi^2} \cos \left(\frac{n\pi}{2} \right) x \right]_{-2}^2$$

$$= \frac{1}{2} \left[\frac{4}{n^2 \pi^2} \left[(2(2) - 1) \cos \frac{n\pi}{2} (2) - (2(-2) - 1) \cos \frac{n\pi}{2} (-2) \right] \right]$$

$$= \frac{2}{n^2 \pi^2} \left[3 \cos n\pi - (-5 \cos n\pi) \right] = \frac{2}{n^2 \pi^2} (3 \cos n\pi + 5 \cos n\pi)$$

$$= \frac{2}{n^2 \pi^2} (8 \cos n\pi) = \frac{16 \cos n\pi}{n^2 \pi^2}$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \left(\frac{n\pi}{l} \right) x dx = \frac{1}{2} \int_{-2}^2 (x^2 - x) \sin \left(\frac{n\pi}{2} \right) x dx$$

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4) $f(x) = \frac{\pi-x}{2}$ in $0 < x < 2$

Sol:-

$$2l = 2 \Rightarrow l = 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l} \int_0^2 f(x) dx = \frac{1}{1} \int_0^2 \frac{\pi-x}{2} dx$

$$= \frac{\pi}{2} \left[x \right]_0^2 - \left[\frac{1}{2} \left(\frac{x^2}{2} \right) \right]_0^2$$

$$= \frac{\pi}{2} (2-0) - \left[\frac{4}{4} - 0 \right]$$

$$a_0 = \pi - 1$$

$$a_n = \frac{1}{l} \int_0^2 f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{1}{1} \int_0^2 \left(\frac{\pi-x}{2}\right) \cos\left(\frac{n\pi x}{1}\right) dx$$

$$= \int_0^2 \frac{\pi-x}{2} \cos(n\pi x) dx$$

$$= \left[\frac{\pi-x}{2} \left(\frac{\sin(n\pi x)}{n\pi} \right) - \left(\frac{-1}{2} \right) \left(\frac{-\cos(n\pi x)}{(n\pi)^2} \right) \right]_0^2$$

$$= \frac{1}{2} \left(\frac{-\cos 2n\pi}{n^2\pi^2} \right) - \frac{1}{2} \left(\frac{-\cos 0}{n^2\pi^2} \right)$$

$$a_n = -\frac{1}{2n^2\pi^2} [(-1)^{2n} - 1] = -\frac{1}{2} [1-1] \left(\frac{1}{n^2\pi^2} \right) = 0$$

$$b_n = \frac{1}{l} \int_0^2 f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \int_0^2 \frac{\pi-x}{2} \left(\sin(n\pi x) \right) dx$$

$$= \left[\frac{\pi-x}{2} \left(\frac{-\cos(n\pi x)}{n\pi} \right) - \left(\frac{-1}{2} \right) \left(\frac{-\sin(n\pi x)}{n^2\pi^2} \right) \right]_0^2$$

$$= \left[\frac{\pi-2}{2n\pi} (-\cos(2\pi n) + \cos 0) + \frac{1}{2} \right]$$

$$= \frac{\pi-2}{2n\pi} ((-1)^{2n} + 1) = \frac{\pi-2}{2n\pi} (2) =$$

$$f(x) = \begin{cases} 2-x & 0 \leq x \leq 4 \\ x-6 & 4 \leq x \leq 8 \end{cases} \text{ in } (0,8)$$

$$2l=8 \Rightarrow l=4$$

$$\begin{aligned} a_0 &= \frac{1}{l} \int_0^8 f(x) dx = \frac{1}{4} \left[\int_0^4 (2-x) dx + \int_4^8 (x-6) dx \right] \\ &= \frac{1}{4} \left[\left(2x - \frac{x^2}{2} \right)_0^4 + \left(\frac{x^2}{2} - 6x \right)_4^8 \right] \\ &= \frac{1}{4} \left[(8-8) - (0) + 32 - 48 - 8 - 24 \right] \end{aligned}$$

$$a_0 = 0$$

$$\begin{aligned} a_n &= \frac{1}{l} \int_0^8 f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{1}{4} \left[\int_0^4 (2-x) \cos\left(\frac{n\pi x}{4}\right) dx \right. \\ &\quad \left. + \int_4^8 (x-6) \cos\left(\frac{n\pi x}{4}\right) dx \right] \\ &= \frac{1}{4} \left[(2-x) \frac{\sin(n\pi x/4)}{(n\pi/4)} - (-1) \frac{(-\cos(n\pi x/4))}{(n\pi/4)^2} \right]_0^4 + \end{aligned}$$

$$\begin{aligned} &\left[(x-6) \frac{\sin(n\pi x/4)}{(n\pi/4)} - (-\cos(n\pi x/4)) \frac{1}{(n\pi/4)^2} \right]_4^8 \\ &= \frac{1}{4} \left[\frac{-16}{n^2\pi^2} (\cos n\pi - \cos 0) + \frac{16}{n^2\pi^2} (\cos 2n\pi - \cos n\pi) \right] \\ &= \frac{1}{4} \left(\frac{16}{n^2\pi^2} \right) [-\cos n\pi + 1 + \cos 2n\pi - \cos n\pi] \end{aligned}$$

$$= \frac{4}{n^2\pi^2} [1+1-2\cos n\pi] = \frac{4}{n^2\pi^2} (2-2\cos n\pi)$$

$$a_n = \frac{8}{n^2\pi^2} (1-\cos n\pi)$$

$$\begin{aligned} b_n &= \frac{1}{l} \int_0^8 f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{1}{4} \left[\int_0^4 (2-x) \sin\left(\frac{n\pi x}{4}\right) dx \right. \\ &\quad \left. + \int_4^8 (x-6) \sin\left(\frac{n\pi x}{4}\right) dx \right] \end{aligned}$$

$$b_n = \frac{1}{4} \left[\frac{(2-n) \cos(n\pi/4) - (-1) (-\sin(n\pi/4))}{(n\pi/4)} \right]_0^4 + \left[\frac{(2-n) \cos(n\pi/4) - (-1) (-\sin(n\pi/4))}{(n\pi/4)} \right]_4^8$$

$$b_n = 0$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2 \pi^2} (1 - \cos n\pi) \cos\left(\frac{n\pi x}{4}\right) //$$

6) $(2x - x^2)$ in $(0, 3)$

Sol:- $2l = 3 \Rightarrow l = \frac{3}{2}$

$$a_0 = \frac{1}{l} \int_0^l f(x) dx = \frac{2}{3} \int_0^3 (2x - x^2) dx = \frac{2}{3} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \frac{2}{3} [9 - 9] = 0 //$$

$$a_n = \frac{1}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{2}{3} \int_0^3 (2x - x^2) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3} \left[\frac{(2x - x^2) \sin(2n\pi x/3)}{2n\pi/3} - (2 - 2x) \frac{(-\cos(2n\pi x/3))}{4n^2\pi^2/9} + \frac{(-2) (-\sin(2n\pi x/3))}{8n^3\pi^3/27} \right]_0^3$$

$$a_n = \frac{2}{3} \left[- (2-6) \left(\frac{-\cos(2n\pi(3)/3)}{4n^2\pi^2} \right) + (2) \left(\frac{-\cos 0}{4n^2\pi^2} \right) \frac{9}{4n^2\pi^2} \right]$$

$$= \frac{2}{3} \left(\frac{9}{4n^2\pi^2} \right) \left[-4(\cos 2n\pi) + 2\cos 0 \right]$$

$$= \frac{-3}{2n^2\pi^2} \left[4\cos 2n\pi + 2\cos 0 \right] = \frac{-3}{2n^2\pi^2} [4 + 2]$$

$$= \frac{-18}{2n^2\pi^2} = \frac{-9}{n^2\pi^2} //$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_0^3 (2x - x^2) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{2}{3} \int_0^3 (2x - x^2) \sin\left(\frac{2n\pi x}{3}\right) dx \\
 &= \frac{2}{3} \left[\frac{(2x - x^2) (-\cos 2n\pi x/3)}{2n\pi/3} - (2 - 2x) \frac{(-\sin 2n\pi x/3)}{4\pi^2 n^2/9} \right. \\
 &\quad \left. + (-2) \frac{(-\cos 2n\pi x/3)}{8n^3\pi^3/27} \right]_0^3 \\
 &= \frac{2}{3} \left[\frac{3(6-9)}{2n\pi} (-\cos 2n\pi + \cos 0) - 2 \left(\frac{27}{8n^3\pi^3} \right) (-\cos 2n\pi + \cos 0) \right] \\
 &= \frac{2}{3} \left[\frac{9}{2n\pi} (1+1) - \frac{27}{n^3\pi^3} (-1+1) \right] \\
 &= \frac{9}{2n\pi} \left(\frac{2}{3} \right) = \frac{3}{n\pi} \\
 \therefore f(x) &= \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\left(\frac{2n\pi x}{3}\right)
 \end{aligned}$$

FOURIER TRANSFORMS

The Fourier transform of a function $f(x)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

where s is the parameter of transformation.

The Inverse Fourier transform of $F(s)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

We also define Fourier sine & cosine transforms by

$$(1) F_s(s) = \int_0^{\infty} f(x) \sin sx \, dx$$

$$(2) F_c(s) = \int_0^{\infty} f(x) \cos sx \, dx$$

Inverse Fourier sine & cosine transforms -

$$(1) f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx \, ds \quad (2) f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx \, ds$$

1) Find the Fourier transform of $f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$

Sol:-

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = 0 + \int_0^{\infty} xe^{-x} e^{isx} dx$$

$$= \int_0^{\infty} x e^{(-1+is)x} dx = \left[\frac{x e^{(-1+is)x}}{(-1+is)} - \frac{(1) e^{(-1+is)x}}{(-1+is)^2} \right]_0^{\infty}$$

$$= \left[\frac{x e^{-(1-is)x}}{(-1+is)} - \frac{e^{-(1-is)x}}{(-1+is)^2} \right]_0^{\infty}$$

$$= \left[\frac{(\infty) e^{-(1-is)\infty}}{-1+is} - \frac{(0)}{(-1+is)} - \frac{(1) - e^{-(1-is)\infty}}{(-1+is)^2} + \frac{e^0}{(-1+is)^2} \right]$$

$$F(s) = \frac{1}{(-1+is)^2}$$

2) $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$

Sol:-

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_0^{\pi} \sin x e^{isx} dx$$

$$= \left[\frac{e^{isx}}{i s^2 + 1} (is \sin x - \cos x) \right]_0^{\pi}$$

$$= \left[\frac{e^{is\pi}}{-s^2 + 1} (is(\sin \pi) - \cos \pi) - \frac{e^0}{-s^2 + 1} (is \sin 0 - \cos 0) \right]$$

$$= \frac{e^{is\pi}}{-s^2 + 1} (1) - \frac{1}{-s^2 + 1} (-1)$$

$$= \frac{e^{is\pi}}{(-s^2 + 1)} + \frac{1}{(s^2 + 1)} = \frac{e^{is\pi} + 1}{(-s^2 + 1)}$$

3) $f(x) = \begin{cases} x & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$

Sol:-

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \int_{-a}^a x e^{isx} dx$$

$$= \left[\frac{x e^{isx}}{is} - \frac{e^{isx}}{is^2} \right]_a^{\infty} = \left[\frac{a e^{isa}}{is} - \frac{(a) e^{-isa}}{is} - \frac{e^{isa}}{(-s^2)} + \frac{e^{-isa}}{(-s^2)} \right]$$

$$= \frac{1}{is} (a e^{isa} + a e^{-isa}) + \frac{1}{s^2} (e^{isa} - e^{-isa})$$

$$= \frac{a}{is} (e^{isa} + e^{-isa}) + \frac{1}{s^2} (e^{isa} - e^{-isa})$$

$$e^{iax} + e^{-iax} = 2 \cos ax$$

$$\frac{e^{iax} - e^{-iax}}{2i} = \sin ax \Rightarrow e^{iax} - e^{-iax} = 2i \sin ax$$

$$= \frac{a}{is} 2 \cos(as) + \frac{1}{s^2} 2i \sin(as)$$

$$F(s) = \frac{2i}{s^2} (\sin(as) - as \cos(as))$$

$$f(x) = e^{-|x|}$$

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\begin{aligned} e^{-|x|} \text{ for } x > 0 &= e^{-x} \\ e^{-|x|} \text{ for } x < 0 &= e^x \end{aligned}$$

$$= \int_{-\infty}^0 e^x e^{isx} dx + \int_0^{\infty} e^{-x} e^{isx} dx$$

$$= \int_{-\infty}^0 e^{(1+is)x} dx + \int_0^{\infty} e^{-(1-is)x} dx$$

$$= \left[\frac{e^{(1+is)x}}{(1+is)} \right]_{-\infty}^0 + \left[\frac{e^{-(1-is)x}}{-(1-is)} \right]_0^{\infty}$$

$$= \left[\frac{e^0}{1+is} - e^{-\infty} \right] + \left[\frac{e^{-(1-is)\infty}}{-(1-is)} - \frac{e^0}{-(1-is)} \right]$$

$$F(s) = \frac{1}{(1+is)} + \frac{1}{(1-is)}$$

$$= \frac{1-is+1+is}{1-i^2s^2} = \frac{2}{1+s^2}$$

* Properties of Fourier Transforms -

- (1) Linearity property - If $f(x)$ & $g(x)$ are 2 functions and $F(s)$ and $G(s)$ are their Fourier transforms then,

$$\boxed{F[a f(x) + b g(x)] = a F(s) + b G(s)}$$

- (2) change of scale : $\boxed{F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)}$ for $a \neq 0$

- (3) Shifting property : (a) $\boxed{F[f(x-a)] = e^{isa} F(s)}$

(b) $\boxed{F[e^{iax} f(x)] = F(s+a)}$

- (4) Modulation property - If $F(s)$ is the Fourier transform of function $f(x)$ then,

$$\boxed{F[f(x) \cos ax] = \frac{1}{2} [F(s+a) + F(s-a)]}$$

- (5) Transform of the derivative - If $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$ then,

$$\boxed{F[f'(x)] = -is F(s)}$$

Problems:

- 5) Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$ and hence

evaluate (a) $\int_0^\infty \frac{x \cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

(b) $\int_0^\infty \frac{x \cos x - \sin x}{x^3} dx$

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-isx} dx$$

$$= \int_{-1}^1 (1-x^2) e^{isx} dx$$

$$= \left[(1-x^2) \frac{e^{isx}}{is} - (-2x) \frac{e^{isx}}{i^2 s^2} + (-2) \frac{e^{isx}}{i^3 s^3} \right]_{-1}^1$$

$$= (0-0) + \left[\frac{2(1)e^{is}}{i^2 s^2} - 2(-1) \frac{e^{-is}}{i^2 s^2} - \frac{2e^{is}}{i^3 s^3} - \frac{2e^{-is}}{i^3 s^3} \right]$$

$$= \frac{-2}{s^2} (e^{is} + e^{-is}) - \frac{2}{-is^3} (e^{is} - e^{-is})$$

$$= \frac{-2}{s^2} (e^{is} + e^{-is}) + \frac{2}{s^3} \left(\frac{e^{is} - e^{-is}}{i} \right)$$

$$= \frac{-2}{s^2} (2\cos s) + \frac{2}{s^3} (2\sin s)$$

$$F(s) = \frac{-4}{s^2} \cos s + \frac{4}{s^3} \sin s = \frac{-4s \cos s + 4 \sin s}{s^3}$$

$$F(s) = \frac{4 \sin s - 4s \cos s}{s^3} = \frac{-4}{s^3} [s \cos s - \sin s]$$

$$\neq \frac{-4}{s^3} (s \cos s - 2 \sin s)$$

We have inverse Fourier transform as, $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$

$$\left\{ \begin{array}{ll} 1-x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{array} \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \quad \text{--- (1)}$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -4 \left(\frac{s \cos s - \sin s}{s^3} \right) e^{-isx} ds$$

8) To evaluate, put $x = \frac{1}{2}$ in eq. (1)

$$\text{at } x = \frac{1}{2}, f(x) = 1-x^2 = \frac{1-1}{4} = \frac{3}{4}$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

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$$\Rightarrow \frac{3}{4} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -4 \left(\frac{s \cos s - \sin s}{s^3} \right) e^{-is/2} ds$$

$$\Rightarrow \frac{3}{4} \left(\frac{2\pi}{-4} \right) = \int_{-\infty}^{\infty} \frac{s \cos s - \sin s}{s^3} \left(\cos \left(\frac{s}{2} \right) - i \sin \left(\frac{s}{2} \right) \right) ds$$

$$\Rightarrow \frac{-3\pi}{8} = \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) \cos \left(\frac{s}{2} \right) ds - i \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) \sin \left(\frac{s}{2} \right) ds$$

$$= + \left(\frac{s \cos s - \sin s}{s^3} \right) \sin \left(\frac{s}{2} \right) = -f(x)$$

NOTE:

In the first part (I) of above integral, when $s = -s$, the function is even. Then, $\int_{-\infty}^{\infty}$ will become $2 \int_0^{\infty}$

In the second part (II) of above integral, when $s = -s$, the function is odd $\Rightarrow$ it is 0.

$$\Rightarrow \frac{-3\pi}{8} = 2 \int_0^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) \cos \left(\frac{s}{2} \right) ds - i(0)$$

$$\Rightarrow \int_0^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) \cos \left(\frac{s}{2} \right) ds = \frac{-3\pi}{16}$$

Replacing s by x in above integral,

$$\therefore \int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \left(\frac{x}{2} \right) dx = \frac{-3\pi}{16}$$

(b) To evaluate, put $x=0$ in eq. (1),

$$\text{at } x=0, f(x) = 1-x^2 = 1-0 = 1$$

$$\Rightarrow 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} -4 \left(\frac{s \cos s - \sin s}{s^3} \right) e^{-is(0)} ds$$

$$\Rightarrow 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} -4 \left(\frac{s \cos s - \sin s}{s^3} \right) ds$$

$$\Rightarrow \frac{2\pi}{-4} = \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) ds$$

$$\Rightarrow \frac{-\pi}{2} = \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) ds$$

Put $s = -s \Rightarrow f(x) = \frac{-s \cos s + \sin s}{-s^3} = \frac{s \cos s + \sin s}{s^3} = f(x)$

The above integral is an even function.

$$\therefore \Rightarrow \frac{-\pi}{2} = 2 \int_0^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) ds$$

$$\Rightarrow \frac{-\pi}{4} = \int_0^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) ds$$

$$\therefore \int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) dx = -\frac{\pi}{4}$$

6) Find Fourier transform of $f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$ $a > 0$ & hence evaluate

(i) $\int_{-\infty}^{\infty} \frac{\sin ax \cos x}{x} dx$ (ii) $\int_0^{\infty} \frac{\sin ax}{x} dx$

sol: Fourier transform $F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-a}^a (1) e^{isx} dx$
 $= \left[\frac{e^{isx}}{is} \right]_{-a}^a = \frac{1}{is} (e^{isa} - e^{-isa}) = \frac{2 \sin(as)}{s}$

$$\therefore F(s) = \frac{2 \sin(as)}{s} \text{ for } s \neq 0$$

as $s \rightarrow 0$, $F[f(x)] = \int_{-a}^a 1 e^{i0x} dx = [x]_{-a}^a$
 $= a - (-a)$
 $= 2a$

Hence $F(s) = \begin{cases} \frac{2 \sin as}{s} & \text{for } s \neq 0 \\ 2a & \text{for } s \rightarrow 0 \end{cases}$

Now, Inverse fourier transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$\Rightarrow f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin as}{s} e^{-isx} ds$$

$$\Rightarrow f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} e^{-isx} ds \quad \text{--- (1)}$$

(i) To find (i),

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx) ds$$

$$\pi f(x) = \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx ds - i \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin sx ds$$

odd function

$$\pi f(x) = \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx ds - 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx ds = \pi f(x)$$

$$= \pi \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

$$= \pi \begin{cases} \pi & |x| \leq a \\ 0 & |x| > a \end{cases}$$

Replace s with x & x with a in the above equation,

$$\int_{-\infty}^{\infty} \frac{\sin ax \cos ax}{x} dx = \begin{cases} \pi, & |a| \leq a \\ 0, & |a| > a \end{cases}$$

(ii) To evaluate (ii), put $x=0$ in (i),

$$0 \in (-a, a) \Rightarrow 1 = f(x)$$

$$f(x) = f(0) = 1$$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} ds$$

Put $s = -s \Rightarrow f \sin as = f(x)$
 $f s$

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin as}{s} ds$$

$\therefore$ even function

$$\therefore \int_0^{\infty} \frac{\sin as}{s} ds = \frac{\pi}{2}$$

Replacing s with x ,

$$\therefore \int_0^{\infty} \frac{\sin ax}{x} = \frac{\pi}{2}$$

*1) Prove that $F\left[f(x)\left(1+\cos\left(\frac{\pi x}{a}\right)\right)\right] = \frac{2\pi^2 \sin as}{s(\pi^2 - a^2 s^2)}$

Sol: Now, $F\left[f(x)\left(1+\cos\left(\frac{\pi x}{a}\right)\right)\right] = F[f(x)] + F\left[f(x)\cos\left(\frac{\pi x}{a}\right)\right]$

$$= \frac{2 \sin as}{s} + \frac{1}{2} \left[F\left(s + \frac{\pi}{a}\right) + F\left(s - \frac{\pi}{a}\right) \right]$$

$$= \frac{2 \sin as}{s} + \frac{1}{2} \left[\frac{2 \sin\left(a\left(s + \frac{\pi}{a}\right)\right)}{\left(s + \frac{\pi}{a}\right)} + \frac{2 \sin\left(a\left(s - \frac{\pi}{a}\right)\right)}{\left(s - \frac{\pi}{a}\right)} \right]$$

$$\begin{aligned} \frac{\sin(as - \pi)}{\sin(-(\pi - as))} &= \frac{2 \sin as}{s} + \frac{a \sin(as + \pi)}{as + \pi} + \frac{a \sin(as - \pi)}{as - \pi} \\ \frac{-\sin(\pi - as)}{-\sin(as)} & \end{aligned}$$

$$= \frac{2 \sin as}{s} - \frac{a}{as + \pi} \sin(\pi + as) + \frac{a}{as - \pi} (-\sin(as))$$

$$= \frac{2 \sin as}{s} - \frac{a \sin(as)}{as + \pi} - \frac{a \sin(as)}{as - \pi}$$

$$= \sin as \left[\frac{2}{s} - \frac{a}{as + \pi} - \frac{a}{as - \pi} \right]$$

$$= \sin as \left[\frac{2(as + \pi)(as - \pi) - as(as - \pi) - as(as + \pi)}{s(as + \pi)(as - \pi)} \right]$$

$$= \frac{\sin as}{s(as + \pi)(as - \pi)} (-2\pi^2) = \frac{-2\pi^2 \sin as}{s(\pi^2 - a^2 s^2)}$$

Hence proved.

2) Find F.T. of $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ & hence deduce $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$

$$= \frac{\pi}{2}$$

Sol: $F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$ $|x| \leq 1$
 $\Rightarrow -1 < x < 1$

$= \int_{-1}^1 f(x) e^{isx} dx$ $-1 \leq x \leq 0 \rightarrow 1 - |x|$
 $= 1 - (-x)$
 $= 1 + x$

$= \int_{-1}^0 f(x) e^{isx} dx + \int_0^1 f(x) e^{isx} dx$ $0 \leq x \leq 1 \rightarrow 1 - |x|$
 $= 1 - x$

$= \int_{-1}^0 (1+x) e^{isx} dx + \int_0^1 (1-x) e^{isx} dx$ $= 1 - x$

$= \left[\frac{(1+x) e^{isx}}{is} - (1) \left(\frac{e^{isx}}{i^2 s^2} \right) \right]_{-1}^0 + \left[\frac{(1-x) e^{isx}}{is} + (1) \left(\frac{e^{isx}}{i^2 s^2} \right) \right]_0^1$

$= \left[\frac{(1) e^0 - (0)}{is} - \left(\frac{e^0}{i^2 s^2} - \frac{e^{-is}}{i^2 s^2} \right) \right] + \left[\frac{0 - 1 e^0}{is} + \left(\frac{0 - 1 e^0}{i^2 s^2} + \frac{e^{is}}{i^2 s^2} \right) \right]$

$= \frac{e^{is} - e^0}{i^2 s^2} - \frac{e^0}{i^2 s^2} + \frac{e^{-is}}{i^2 s^2} - \frac{1}{is} + \frac{e^{is}}{i^2 s^2} - \frac{1}{i^2 s^2}$

$= \left(\frac{e^{is}}{i^2 s^2} + \frac{e^{-is}}{i^2 s^2} \right) - \frac{2}{i^2 s^2} = \frac{2 \cos s}{-s^2} - \frac{2}{-s^2} = \frac{2}{s^2} (1 - \cos s)$

By Inverse Fourier transform,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{s^2} (1 - \cos s) e^{-isx} ds$$

Put $x=0$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos s}{s^2} ds$$

Put $s = -s \Rightarrow \frac{1 - \cos s}{s^2}$

← even function

$$1 = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos s}{s^2} ds$$

$$\therefore \int_0^{\infty} \frac{1 - \cos s}{s^2} ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{2 \sin^2(s/2)}{s^2} ds = \frac{\pi}{2}$$

$$\text{Put } \frac{s}{2} = t \Rightarrow s = 2t$$

$$ds = 2dt$$

$$\Rightarrow \int_0^{\infty} \frac{2 \sin^2 t}{4t^2} (2dt) = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$$

$$\text{Replacing } t \text{ by } x \Rightarrow \int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

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Q) Find the F.T. of $e^{-a^2 x^2}$, $a > 0$ & hence deduce that $e^{-x^2/2}$ is self reciprocal w.r.t. F.T.

$$\text{Sol:- } F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx = \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx$$

$$F(s) = \int_{-\infty}^{\infty} e^{-(a^2 x^2 - isx)} dx$$

$$\text{NOTE: } a^2 x^2 - isx = (ax)^2 - 2(ax)\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2 = \left[\frac{ax - is}{2a}\right]^2 + \frac{s^2}{4a^2}$$

$$\Rightarrow F(s) = \int_{-\infty}^{\infty} e^{-\left[\frac{ax - is}{2a}\right]^2 + \frac{s^2}{4a^2}} dx$$

$$= e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left[\frac{ax - is}{2a}\right]^2} dx$$

$$= e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$F(s) = \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$\frac{ax - is}{2a} = t$$

$$a dx = dt$$

$$dx = \frac{dt}{a}$$

$$\text{NOTE: } \int_{-\infty}^{\infty} e^{-t^2} dt = \Gamma(1/2) = \sqrt{\pi}$$

$$\therefore F(s) = \frac{e^{-s^2/4a^2}}{a} [\sqrt{\pi}]$$

$$\Rightarrow F(s) = \frac{\sqrt{\pi}}{a} e^{-(s/2a)^2}$$

$$\text{Put } a^2 = \frac{1}{2} \Rightarrow F(s) = F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(\frac{1}{\sqrt{2}})} e^{-s^2/4a^2}$$

$$a = \frac{1}{\sqrt{2}}$$

$$F(e^{-x^2/2}) = \sqrt{2\pi} e^{-s^2/4(\frac{1}{\sqrt{2}})^2}$$

$$= \sqrt{2\pi} e^{-s^2/2}$$

$$= \sqrt{2\pi} e^{-s^2/2}$$

$$\therefore F(e^{-x^2/2}) = \sqrt{2\pi} e^{-s^2/2}$$

$$\Rightarrow F[e^{-x^2/2}] \text{ is constant times } e^{-s^2/2}$$

$\therefore$ It is self reciprocal w.r.t. Fourier Transform.

b) Given $F[e^{-x^2}] = \sqrt{\pi} e^{-s^2/4}$, find F.T. of

(a) $e^{-x^2} \cos 3x$

(b) $e^{-x^2/3}$

(c) $e^{-4(x-3)^2}$

(d) $e^{-2(x-3)^2}$

Sol:- a)

By modulation property,

$$F[f(x) \cos ax] = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$\Rightarrow F[e^{-x^2} (\cos 3x)] = \frac{1}{2} [F(s+3) + F(s-3)]$$

$$= \frac{1}{2} [\sqrt{\pi} e^{-(s+3)^2/4} + \sqrt{\pi} e^{-(s-3)^2/4}]$$

$$= \frac{\sqrt{\pi}}{2} (e^{-(s+3)^2/4} + e^{-(s-3)^2/4})$$

(b) By change of scale, $F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

$$F[e^{-x^2/3}] = F[e^{-(x/\sqrt{3})^2}] \text{ where } a = \frac{1}{\sqrt{3}}$$

$$\begin{aligned}
 &= \frac{1}{\left(\frac{1}{\sqrt{3}}\right)} F\left[\frac{s}{\frac{1}{\sqrt{3}}}\right] = \sqrt{3} F(\sqrt{3}s) \\
 &= \sqrt{3} \left[\sqrt{\pi} e^{-\frac{(\sqrt{3}s)^2}{4}} \right] \\
 &= \sqrt{3\pi} e^{-\frac{3s^2}{4}}
 \end{aligned}$$

(c) $F[e^{-4(x-3)^2}] \Rightarrow$ we have $F[f(x-a)] = e^{ias} F(s)$ (shifting property)
 $= e^{i3s} F[e^{-4x^2}]$ — ①

$$\begin{aligned}
 F[e^{-4x^2}] &= F[e^{-(2x)^2}] = \frac{1}{2} F\left(\frac{s}{2}\right) \quad \text{where } a=2 \\
 &= \frac{\sqrt{\pi}}{2} e^{-\frac{(s/2)^2}{4}} \quad \text{change of scale} = \frac{1}{a} F(s/a)
 \end{aligned}$$

$$F[e^{-4x^2}] = \frac{\sqrt{\pi}}{2} e^{-s^2/16} \quad \text{--- ②}$$

$$\begin{aligned}
 \text{② in ①} \Rightarrow F[e^{-4(x-3)^2}] &= e^{i3s} \left(\frac{\sqrt{\pi}}{2} e^{-s^2/16} \right) \\
 &= \frac{\sqrt{\pi}}{2} e^{(3is - s^2/16)}
 \end{aligned}$$

(d) $F[e^{-2(x-3)^2}] \Rightarrow$ we have $F[f(x-a)] = e^{ias} F(s)$
 $= e^{i3s} F[e^{-2x^2}]$ — ①

$$\begin{aligned}
 F[e^{-2x^2}] &= F[e^{-(\sqrt{2}x)^2}] = \frac{1}{\sqrt{2}} F\left(\frac{s}{\sqrt{2}}\right) \\
 &= \frac{\sqrt{\pi}}{\sqrt{2}} e^{-\frac{(\sqrt{2}s)^2}{4}} = \sqrt{\frac{\pi}{2}} e^{-s^2/8} \quad \text{--- ②}
 \end{aligned}$$

$$\begin{aligned}
 \text{② in ①, } F[e^{-2(x-3)^2}] &= e^{i3s} \left(\sqrt{\frac{\pi}{2}} e^{-s^2/8} \right) \\
 &= \sqrt{\frac{\pi}{2}} e^{(3is - s^2/8)}
 \end{aligned}$$

10) Find the Inverse F.T. of ~~$e^{-a|s|}$~~ $e^{-a|s|}$

Sol:-

$$\begin{aligned}
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds \quad |s| \Rightarrow \begin{matrix} -s & s > 0 \\ -s & s < 0 \end{matrix} \\
 &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{-a|s|} e^{-isx} ds + \int_0^{\infty} e^{-a|s|} e^{-isx} ds \right] \\
 &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{(as-isx)} ds + \int_0^{\infty} e^{(-as-isx)} ds \right] \\
 &= \frac{1}{2\pi} \left[\int_{-\infty}^0 e^{(a-ix)s} ds + \int_0^{\infty} e^{-(a+ix)s} ds \right] \\
 &= \frac{1}{2\pi} \left[\frac{e^{(a-ix)s}}{(a-ix)} \right]_{-\infty}^0 + \left[\frac{e^{-(a+ix)s}}{-(a+ix)} \right]_0^{\infty} \\
 &= \frac{1}{2\pi} \left[0 - \frac{1}{(a-ix)} + \frac{e^{-\infty} - e^0}{-(a+ix)} \right] \\
 &= \frac{1}{2\pi} \left[-\frac{1}{a-ix} + \frac{1}{a+ix} \right] \\
 &= \frac{1}{2\pi} \left[\frac{a+ix+a-ix}{a^2 - i^2x^2} \right] = \frac{2a}{2\pi(a^2+x^2)}
 \end{aligned}$$

$$f(x) = \frac{a}{\pi(a^2+x^2)}$$

11) Find the Inverse F.T. of e^{-s^2}

Sol:-

$$\begin{aligned}
 f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-s^2} e^{-isx} ds \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(s^2+isx)} ds
 \end{aligned}$$

NOTE:

$$s^2 + isx = s^2 + 2(s)\left(\frac{ix}{2}\right) + \left(\frac{ix}{2}\right)^2 - \left(\frac{ix}{2}\right)^2 = \left(\frac{s+ix}{2}\right)^2 + \left(\frac{x}{2}\right)^2$$

$$\begin{aligned}
 \Rightarrow f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{s+ix}{2}\right)^2 + \frac{x^2}{4}\right]} ds \\
 &= \frac{1}{2\pi} e^{-x^2/4} \int_{-\infty}^{\infty} e^{-\left[\frac{s+ix}{2}\right]^2} ds
 \end{aligned}$$

$$\text{Put } \frac{s+ix}{2} = t \Rightarrow ds = dt$$

$$f(x) = \frac{1}{2\pi} e^{-x^2/4} \int_{-\infty}^{\infty} e^{-t^2} dt$$

NOTE - $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ or $2 \int_0^{\infty} e^{-t^2} dt = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$

$$\Rightarrow f(x) = \frac{1}{2\pi} e^{-x^2/4} (\sqrt{\pi})$$

$$f(x) = \frac{e^{-x^2/4}}{2\sqrt{\pi}}$$

* Fourier Cosine Transform -

13) Find F.C.T of e^{-x^2}

Sol:- $F(s) = \int_0^{\infty} f(x) \cos sx \, dx = \int_0^{\infty} e^{-x^2} \cos sx \, dx$

NOTE - Leibnitz Rule of differentiation under Integral sign -

If $f(x, \alpha)$ and $\frac{\partial f}{\partial \alpha}(x, \alpha)$ is continuous function of x & α

then, $\frac{d}{d\alpha} \int_a^b f(x, \alpha) \, dx = \int_a^b \left(\frac{\partial}{\partial \alpha} f(x, \alpha) \right) \, dx$

Let $I = \int_0^{\infty} e^{-x^2} \cos sx \, dx$

$$\frac{d}{ds}(I) = \frac{d}{ds} \left[\int_0^{\infty} e^{-x^2} \cos sx \, dx \right] = \int_0^{\infty} \frac{\partial}{\partial s} (e^{-x^2} \cos sx \, dx)$$

$$= \int_0^{\infty} e^{-x^2} (-\sin sx) \, dx$$

$$= \int_0^{\infty} e^{-x^2} (-\sin sx) x \, dx$$

$$e^{-x^2} = t$$

$$e^{-x^2} (-2x \, dx) = dt$$

$$\frac{dI}{ds} = \frac{1}{2} \int_0^{\infty} \sin sx (-2x e^{-x^2}) \, dx$$

$$\Rightarrow \frac{dI}{ds} = \frac{1}{2} \int_0^{\infty} \sin sx \, d(e^{-x^2}) = \left[\frac{1}{2} \sin sx (e^{-x^2}) \right]_0^{\infty} - \frac{1}{2} \int_0^{\infty} e^{-x^2} \cos sx \, dx$$

$$\frac{dI}{ds} = 0 - \frac{1}{2} \left(\int_0^{\infty} e^{-x^2} \cos sx \, dx \right)$$

$$\frac{dI}{ds} = -\frac{s}{2} (I)$$

$$\Rightarrow \frac{dI}{ds} = -\frac{s}{2} ds$$

Integrating on both sides,

$$\log I = -\frac{1}{2} \frac{s^2}{2} + \log C$$

$$\log I - \log C = -\frac{s^2}{4}$$

$$\log \left(\frac{I}{C} \right) = -\frac{s^2}{4} \Rightarrow \frac{I}{C} = e^{-s^2/4}$$

$$\therefore I = C e^{-s^2/4}$$

To find C $\Rightarrow$ put $s=0$

$$I = C e^0 = C$$

$$\text{Now } I = \int_0^\infty e^{-x^2} \cos sx \, dx \quad \{\text{at } s=0\}$$

$$\text{At } s=0, I = \int_0^\infty e^{-x^2} \, dx$$

$$I = \frac{\sqrt{\pi}}{2} = C$$

$$\text{Hence, } I = C e^{-s^2/4}$$

$$= \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

$$\therefore F_c[e^{-x^2}] = \frac{\sqrt{\pi}}{2} e^{-s^2/4}$$

14) Find F.C.T of e^{-ax} and deduce the fourier cosine transform of $x e^{-ax}$ & hence evaluate $\int_0^\infty \frac{\cos xx}{a^2 + x^2} \, dx$.

Sol:-

$$F_c(f(x)) = \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \left[\frac{e^{-ax}}{(-a^2 + s^2)} [(-a) \cos sx + s \sin sx] \right]_0^\infty$$

$$= \frac{e^{-ax}}{a^2 + s^2} \left[-a \cos s(\infty) + a \cos s(0) + s \right]$$

$$F(s) = 0 + \frac{1}{a^2 + s^2} [-a + 0] = \frac{-a}{a^2 + s^2}$$

$$\therefore F_c[e^{-ax}] = \frac{a}{a^2 + s^2} \quad \text{--- (1)}$$

$$\text{We have } F_c[e^{-ax}] = \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\frac{d}{da} [F_c[e^{-ax}]] = \frac{d}{da} \left[\int_0^\infty e^{-ax} \cos sx \, dx \right]$$

$$\text{By Leibnitz Rule,} \quad = \int_0^\infty \frac{\partial}{\partial a} (e^{-ax} \cos sx) \, dx$$

$$= \int_0^\infty e^{-ax} (-x) \cos sx \, dx$$

$$\frac{d}{da} F_c[e^{-ax}] = - \int_0^\infty x e^{-ax} \cos sx \, dx \quad \text{--- (2)}$$

$$\frac{d}{da} \left[\frac{a}{a^2 + s^2} \right] = \frac{(a^2 + s^2) \cdot 1 - a(2a)}{(a^2 + s^2)^2}$$

$$= \frac{a^2 + s^2 - 2a^2}{(a^2 + s^2)^2} = \frac{s^2 - a^2}{(a^2 + s^2)^2}$$

$$\text{Hence by eq. (2), we have, } \frac{d}{da} [F_c[e^{-ax}]] = -F_c[xe^{-ax}]$$

$$\Rightarrow F_c[xe^{-ax}] = - \left(\frac{s^2 - a^2}{(a^2 + s^2)^2} \right) = \frac{a^2 - s^2}{(a^2 + s^2)^2}$$

to evaluate, By Inverse Fourier ^{Cosine} Transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(s) \cos sx \, ds$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a}{a^2 + s^2} \cos sx \, ds$$

$$\left\{ \text{Take } f(x) = e^{-ax} \Rightarrow F_c(s) = F_c(f(x)) = F_c(e^{-ax}) = \frac{a}{a^2 + s^2} \right\}$$

$$\Rightarrow \frac{\pi}{2a} e^{-ax} = \int_0^\infty \frac{\cos sx}{a^2 + s^2} \, ds$$

$$\text{Put } s = \alpha \Rightarrow \frac{\pi}{2\alpha} e^{-\alpha x} = \int_0^{\infty} \frac{\cos \alpha x}{\alpha^2 + x^2} dx$$

15) Find FCT of $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Sol: $F_c[f(x)] = \int_0^{\infty} f(x) \cos sx \, dx = \int_0^1 x \cos sx \, dx + \int_1^2 (2-x) \cos sx \, dx + 0$

$$= \left[x \sin sx + \frac{1}{s} \cos sx \right]_0^1 + \left[(2-x) \sin sx - \frac{1}{s} \cos sx \right]_1^2$$

$$= \left[\sin s - 0 + \frac{\cos s}{s} - \frac{\cos 0}{s} \right] + \left[0 - 1(\sin s) - \frac{\cos 2s}{s} + \frac{\cos s}{s} \right]$$

$$= \left[\frac{\sin s}{s} - 0 + \frac{\cos s}{s^2} - \frac{1}{s^2} + 0 - \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2} \right]$$

$$= \left[\frac{\sin s}{s} - 0 + \frac{\cos s}{s^2} - \frac{1}{s^2} + 0 - \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2} \right]$$

$$F_c[f(x)] = \frac{2 \cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2} = \frac{2 \cos s - \cos 2s - 1}{s^2}$$

16) $f(x) = e^{-ax} \cos ax$ 17) $f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$

18) $f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$

Sol: $F_c[f(x)] = \int_0^{\infty} f(x) \cos sx \, dx = \int_0^a \cos sx \, dx + 0$

$$= \left[\frac{\sin sx}{s} \right]_0^a = \frac{\sin as}{s}$$

$$17) f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$$

Sol: $F_c[f(x)] = \int_0^a \cos s \cos sx \, dx + 0 = \int_0^a \cos(sx) \cos x \, dx$

$$= \frac{1}{2} \left[\int_0^a (\cos(s+1)x + \cos(s-1)x) \, dx \right]$$

$$= \frac{1}{2} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{(s-1)} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{\sin(s+1)a}{s+1} - 0 + \frac{\sin(s-1)a}{(s-1)} - 0 \right]$$

$$= \frac{1}{2} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{(s-1)} \right] //$$

$$16) f(x) = e^{-ax} \cos ax$$

Sol: $F_c[f(x)] = \int_0^\infty e^{-ax} \cos ax \cos sx \, dx = \int_0^\infty e^{-ax} \frac{1}{2} [\cos(s+a)x + \cos(s-a)x] \, dx$

$$= \frac{1}{2} \int_0^\infty e^{-ax} \cos(s+a)x \, dx + \frac{1}{2} \int_0^\infty e^{-ax} \cos(s-a)x \, dx$$

$$= \frac{1}{2} \left[\frac{e^{-ax}}{a^2 + (s+a)^2} [-a \cos(s+a)x - (s+a) \sin(s+a)x] \right]_0^\infty +$$

$$\frac{1}{2} \left[\frac{e^{-ax}}{a^2 + (s-a)^2} [-a \cos(s-a)x - (s-a) \sin(s-a)x] \right]_0^\infty$$

$$= \frac{1}{2} \left[0 - \frac{1}{a^2 + (s+a)^2} [-a + 0] \right] + \frac{1}{2} \left[0 - \frac{1}{a^2 + (s-a)^2} [-a - 0] \right]$$

$$= \frac{a}{2(a^2 + (s+a)^2)} + \frac{a}{2(a^2 + (s-a)^2)}$$

$$= \frac{a}{2a^2 + 2(s+a)^2} + \frac{a}{2a^2 + 2(s-a)^2} \quad H/A$$

Find Fourier Sine Transform of -

$$18) f(x) = \begin{cases} 4x & 0 < x < 1 \\ 4-x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

$$17) f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x \geq a \end{cases}$$

$$\begin{aligned} \text{sol: } F_c[f(x)] &= \int_0^a \cos x \cos sx \, dx + 0 = \int_0^a \cos(x) \cos sx \, dx \\ &= \frac{1}{2} \left[\int_0^a (\cos(s+1)x + \cos(s-1)x) \, dx \right] \\ &= \frac{1}{2} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{(s-1)} \right]_0^a \\ &= \frac{1}{2} \left[\frac{\sin(s+1)a}{s+1} - 0 + \frac{\sin(s-1)a}{(s-1)} - 0 \right] \\ &= \frac{1}{2} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{(s-1)} \right] // \end{aligned}$$

$$16) f(x) = e^{-ax} \cos ax$$

$$\begin{aligned} \text{sol: } F_c[f(x)] &= \int_0^\infty e^{-ax} \cos ax \cos sx \, dx = \int_0^\infty e^{-ax} \frac{1}{2} [\cos(s+a)x + \cos(s-a)x] \, dx \\ &= \frac{1}{2} \int_0^\infty e^{-ax} \cos(s+a)x \, dx + \frac{1}{2} \int_0^\infty e^{-ax} \cos(s-a)x \, dx \\ &= \frac{1}{2} \left[\frac{e^{-ax}}{a^2 + (s+a)^2} [-a \cos(s+a)x - (s+a) \sin(s+a)x] \right]_0^\infty + \\ &\quad \frac{1}{2} \left[\frac{e^{-ax}}{a^2 + (s-a)^2} [-a \cos(s-a)x - (s-a) \sin(s-a)x] \right]_0^\infty \\ &= \frac{1}{2} \left[0 - \frac{1}{a^2 + (s+a)^2} [-a + 0] \right] + \frac{1}{2} \left[0 - \frac{1}{a^2 + (s-a)^2} [-a + 0] \right] \\ &= \frac{a}{2(a^2 + (s+a)^2)} + \frac{a}{2(a^2 + (s-a)^2)} \\ &= \frac{a}{2a^2 + 2(s+a)^2} + \frac{a}{2a^2 + 2(s-a)^2} // \end{aligned}$$

Find Fourier Sine Transform of -

$$18) f(x) = \begin{cases} 4x & 0 < x < 1 \\ 4-x & 1 < x < 4 \\ 0 & x > 4 \end{cases}$$

Sol: $F_s[f(x)] = \int_0^{\infty} f(x) \sin sx \, dx$

$$\begin{aligned}
 F_s[f(x)] &= \int_0^1 4x \sin sx \, dx + \int_1^4 (4-x) \sin sx \, dx + 0 \\
 &= \left[4x \left(\frac{\cos sx}{s} \right) - 4 \left(\frac{-\sin sx}{s^2} \right) \right]_0^1 + \left[(4-x) \left(\frac{\cos sx}{s} \right) + \left(\frac{-\sin sx}{s^2} \right) \right]_1^4 \\
 &= \left[4 \left(\frac{\cos s}{s} \right) - 0 + \frac{4 \sin s}{s^2} - 0 \right] + \left[0 + \frac{3 \cos s}{s} - \frac{\sin 4s}{s^2} \right. \\
 &\quad \left. + \frac{\sin s}{s^2} \right] \\
 &= -\frac{4 \cos s}{s} + \frac{4 \sin s}{s^2} + \frac{3 \cos s}{s} - \frac{\sin 4s}{s^2} + \frac{\sin s}{s^2} \\
 &= \frac{4 \sin s - \sin 4s}{s^2} + \frac{\sin s}{s^2} - \frac{\cos s}{s} = \frac{5 \sin s - \sin 4s - \cos s}{s^2}
 \end{aligned}$$

2) $f(x) = \begin{cases} \sin x & 0 < x < a \\ 0 & x \geq a \end{cases}$

Sol: $F_s[f(x)] = \int_0^{\infty} f(x) \sin sx \, dx$

$$\begin{aligned}
 F_s[f(x)] &= \int_0^a \sin x \sin sx \, dx + 0 \\
 &= \frac{1}{2} \int_0^a (\cos(s-1)x - \cos(s+1)x) \, dx \\
 &= \left[\frac{1}{2} \frac{\sin(s-1)x}{(s-1)} \right]_0^a - \frac{1}{2} \left[\frac{\sin(s+1)x}{(s+1)} \right]_0^a \\
 &= \frac{1}{2} \left[\frac{\sin(s-1)a}{(s-1)} - 0 \right] - \frac{1}{2} \left[\frac{\sin(s+1)a}{(s+1)} - 0 \right] \\
 &= \frac{1}{2} \left[\frac{\sin(s-1)a}{(s-1)} - \frac{\sin(s+1)a}{(s+1)} \right]
 \end{aligned}$$

$$(3) f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x \geq a \end{cases}$$

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$$(4) \frac{e^{-ax}}{x}, \quad a > 0, \quad x > 0 \text{ \& also deduce that } \int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi}{2} e^{-a}$$

Sol:-

$$f(x) = e^{-ax}$$

$$F_s(s) = \int_0^{\infty} f(x) \sin sx \, dx; I = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx$$

then

$$\frac{dI}{ds} = \int_0^{\infty} \frac{\partial}{\partial s} \left(\frac{e^{-ax}}{x} \sin sx \right) dx$$

$$= \int_0^{\infty} \frac{e^{-ax}}{x} \cos sx \cdot x \cdot dx$$

$$\therefore \frac{dI}{ds} = \int_0^{\infty} e^{-ax} \cos sx \, dx = \frac{e^{-ax}}{a^2 + s^2} \left[-a \cos sx + s \sin sx \right]_0^{\infty}$$

$$= \frac{1}{a^2 + s^2} [0 - 1(-a)] = \frac{a}{a^2 + s^2}$$

$$\frac{dI}{ds} = \frac{a}{a^2 + s^2}$$

$$\Rightarrow I = \int \frac{a}{a^2 + s^2} ds \Rightarrow I = \tan^{-1}\left(\frac{s}{a}\right) + C$$

$$\text{At } s=0, \quad I = \tan^{-1}(0) + C$$

$$I = C$$

$$\text{Also } I = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx, \text{ at } s=0, \quad I=0$$

$$\Rightarrow \boxed{C=0}$$

$$\text{Hence } I = \tan^{-1}\left(\frac{s}{a}\right) \text{ i.e. } F_s(s) = \tan^{-1}\left(\frac{s}{a}\right)$$

Now by Fourier Inverse Sine Transform,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$(3) f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x \geq a \end{cases}$$

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(A) $\frac{e^{-ax}}{x}$, $a > 0$, $x > 0$ & also deduce that $\int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi}{2} e^{-a}$

Sol:-

$$f(x) = e^{-ax}$$

$$F_s(s) = \int_0^{\infty} f(x) \sin sx \, dx; I = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx$$

then

$$\begin{aligned} \frac{dI}{ds} &= \int_0^{\infty} \frac{\partial}{\partial s} \left(\frac{e^{-ax}}{x} \sin sx \right) dx \\ &= \int_0^{\infty} \frac{e^{-ax}}{x} \cos sx \cdot x \cdot dx \end{aligned}$$

$$\begin{aligned} \therefore \frac{dI}{ds} &= \int_0^{\infty} e^{-ax} \cos sx \, dx = \frac{e^{-ax}}{a^2 + s^2} \left[-a \cos sx + s \sin sx \right]_0^{\infty} \\ &= \frac{1}{a^2 + s^2} [0 - 1(-a)] = \frac{a}{a^2 + s^2} \end{aligned}$$

$$\frac{dI}{ds} = \frac{a}{a^2 + s^2}$$

$$\Rightarrow I = \int \frac{a}{a^2 + s^2} ds \Rightarrow I = \tan^{-1}\left(\frac{s}{a}\right) + C$$

$$\text{At } s=0, I = \tan^{-1}(0) + C$$

$$I = C$$

$$\text{Also } I = \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx, \text{ at } s=0, I=0$$

$$\Rightarrow \boxed{C=0}$$

$$\text{Hence } I = \tan^{-1}\left(\frac{s}{a}\right) \text{ i.e. } F_s(s) = \tan^{-1}\left(\frac{s}{a}\right)$$

Now by Fourier Inverse Sine Transform,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \tan^{-1}\left(\frac{s}{a}\right) \sin sx \, ds$$

$$\frac{\pi}{2} \frac{e^{-ax}}{x} = \int_0^{\infty} \tan^{-1}\left(\frac{s}{a}\right) \sin sx \, ds$$

Put $x=1$, $\frac{\pi}{2} e^{-a} = \int_0^{\infty} \tan^{-1}\left(\frac{s}{a}\right) \sin s \, ds$

Now change s to x in the RHS,

$$\int_0^{\infty} \tan^{-1}\left(\frac{x}{a}\right) \sin x \, dx = \frac{\pi e^{-a}}{2} \text{ Hence proved.}$$

- 5) Find Fourier Sine Transform of e^{-ax} , $x > 0$, $a > 0$ and hence evaluate $\int_0^{\infty} \frac{x \sin kx}{a^2 + x^2} \, dx$ & hence obtain the F.S.T. of $\frac{x}{a^2 + x^2}$.

Sol:- $F_s(s) = \int_0^{\infty} f(x) \sin sx \, dx = \int_0^{\infty} e^{-ax} \sin sx \, dx$

$$= \frac{e^{-ax}}{a^2 + s^2} \left[-a \sin sx - s \cos sx \right]_0^{\infty}$$

$$\therefore F_s(s) = \frac{1}{a^2 + s^2} \left[0 - 1(0 - s) \right] = \frac{s}{a^2 + s^2}$$

Now Inverse Fourier Sine Transform is,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F(s) \sin sx \, ds$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$\Rightarrow \frac{\pi}{2} f(x) = \int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$\Rightarrow \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-ax} \int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds$$

change s to x , x to k .

We get, $\int_0^{\infty} \frac{x \sin kx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ak}$

$$\therefore F_s \left[\frac{x}{a^2 + x^2} \right] = \int_0^{\infty} \frac{x}{a^2 + x^2} \sin sx \, dx$$

$$= \frac{\pi}{2} e^{-as} \quad (s=k)$$

6) Find F.S.T. of $e^{-|x|}$ and hence P.T. $\int_0^{\infty} \frac{x}{1+x^2} \sin mx \, dx = \frac{\pi}{2} e^{-m}$, $m > 0$.

Sol: $F_s(s) = \int_0^{\infty} f(x) \sin sx \, dx$

$$= \int_0^{\infty} e^{-x} \sin sx \, dx$$

$e^{-|x|} \rightarrow x > 0$
 $\Rightarrow e^{-x}$

7) Find the Inverse Cosine Fourier transform of $F_c(s) = e^{-as}$

8) Find the I.F.C.T of $F_c(s) = \begin{cases} \frac{1}{2\pi} \left(\frac{a-s}{2} \right) & 0 < s < 2a \\ 0 & s \geq 2a \end{cases}$

$$\begin{aligned} 7) f(x) &= \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx \, ds \\ &= \frac{2}{\pi} \int_0^{\infty} e^{-as} \cos sx \, ds \\ &= \frac{2}{\pi} \frac{e^{-as}}{a^2 + x^2} \left[-a \cos sx + x \sin sx \right]_0^{\infty} \\ &= \frac{2}{\pi} \left[(e^{-\infty}) [\infty] - \frac{e^{-0}}{a^2 + x^2} [-a + x(0)] \right] \end{aligned}$$

$$f(x) = \frac{2}{\pi} \left[\frac{a}{a^2 + x^2} \right]$$

$$\begin{aligned} 8) f(x) &= \frac{2}{\pi} \int_0^{\infty} F_c(s) \cos sx \, ds \\ &= \frac{2}{\pi} \int_0^{2a} \frac{1}{2\pi} \left(\frac{a-s}{2} \right) \cos sx \, ds + 0 \\ &= \frac{2}{\pi} \left(\frac{1}{2\pi} \right) \int_0^{2a} \left(\frac{a-s}{2} \right) \cos sx \, ds \\ &= \frac{1}{\pi^2} \left[\frac{a-s}{2} \frac{\sin sx}{s} - \left(\frac{-1}{2} \right) \left(\frac{-\cos sx}{s^2} \right) \right]_0^{2a} \\ &= \frac{1}{\pi^2} \left[\left(\frac{a-2a}{2} \right) \frac{\sin 2ax}{x} - \left(\frac{a-0}{2} \right) \frac{\sin 0}{x} - \frac{1}{2} \frac{\cos 2ax}{x^2} + \frac{1}{2} \frac{\cos 0}{x^2} \right] \\ &= \frac{1}{\pi^2} \left[\frac{1}{2x^2} - \frac{1}{2} \frac{\cos 2ax}{x^2} \right] \end{aligned}$$

$$f(x) = \frac{1 - \cos 2ax}{2\pi^2 x^2} = \frac{\sin^2(ax)}{\pi^2 x^2}$$

4) Find Inverse F.S.T. of

i) $e^{-as} \cdot \frac{2x}{\pi(a^2+x^2)}$

ii) $\frac{e^{-as}}{s} \cdot \frac{2}{\pi} \tan^{-1}\left(\frac{s}{a}\right)$