Solve $y' = x + y^2$ at $y(0) = 1$ for $x = 0.2$ in 2 steps using Euler's modified method.

$f(x,y) = x + y^2$ $x_0 = 0$, $y_0 = 1$

2 steps $\Rightarrow$ $n = 2$, $x_2 = 0.2 = x_0 + 2h$

$0.2 = 0 + 2h \Rightarrow h = 0.1 \Rightarrow x_1 = 0.1$ $x_2 = 0.2$.

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$y_1^{(k+1)} = y_0 + \frac{h}{2}\{f(x_0, y_0) + f(x_1, y_1^{(k)})\}$ $k = 0, 1$

$y_1^{(0)} = 1 + 0.1(0 + 1^2) = 1.1$

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})\}$

$= 1 + \frac{0.1}{2}[0 + 1^2 + 0.1 + (1.1)^2] = 1.1155.$

$y_1^{(2)} = 1 + \frac{0.1}{2}[0 + 1^2 + 0.1 + (1.1155)^2] = 1.1172.$

let $y_1 = y_1(0.1) \sim 1.1172$.

$y_2^{(0)} = y_1 + hf(x_1, y_1)$

$y_2^{(k+1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(k)})]$

$y_2^{(0)} = 1.1172 + 0.1[0.1 + (1.1172)^2] \simeq 1.2520$

$y_2^{(1)} = 1.1172 + \frac{0.1}{2}[0.1 + (1.1172)^2 + 0.2 + (1.2520)^2]$

$= 1.2729.$

$y_2^{(2)} = 1.27562$

Q) Solve $y' = x + y^2$ at $y(0) = 1$ for $x = \underbrace{0}_{x_0}(\underbrace{0.1}_{h})(\underbrace{0.2}_{x_n})$ using Runge-Kutta method of order 4

$f(x,y) = x + y^2$ $x_0 = 0$ $y_0 = 1$

$x_1 = x_0 + h = 0.1$ $x_2 = x_1 + h = 0.2$

$y_1 = ?$ $y_2 = ?$

$y_1 = y_0 + k$ $\qquad k = \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$ $\qquad k = \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$

$k_1 = 0.1(0 + 1) = 0.1$

$k_2 = (0.1) f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1 f(0.05, 1.05)$

$= (0.1)\{0.05 + (1.05)^2\}$

$k_2 = 0.11525$

$k_3 = 0.1 f\left(0.05, 1 + \frac{0.11525}{2}\right) = 0.1 f(0.05, 1.0576)$

$= 0.1 \{0.05 + 1.0576^2\} = 0.11685$

$k_4 = 0.1 f(0.1, 1.11685)$

$= 0.1\{0.1 + 1.11685^2\} = 0.13473$

$y_1 = y_0 + \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6} = 1.11649$

$x_1 = 0.1, \quad y_1 = 1.11649$

$y_2 = y_1 + \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_1 = hf(x_1, y_1)$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$ $\qquad k = \dfrac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$

$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$

$k_1 = 0.1(0.1 + 1.11649^2) = 0.13465$

$k_2 = 0.1 f\left(0.1 + \frac{0.1}{2}, 1.11649 + \frac{0.13465}{2}\right) = 0.1 f(0.15, 1.183815)$

$= (0.1)(0.15 + (1.183815)^2) = 0.15514.$

$$K_3 = 0.1\, f\left(0.1 + \frac{0.1}{2},\ 1.11649 + \frac{0.15514}{2}\right)$$

$$= \cancel{0.1\, f(0.15 +}\,(1.57578) = 0.1 \times 1.57578 = 0.157578$$

$$K_4 = 0.1\, f(0.2,\ 1.11649 + 0.15758) = 0.1\left(0.2 + (1.11649 + 0.15758)^2\right)$$

$$= 0.18232$$

$$y_2 = y_1 + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6} = 1.273557$$

Solve $y' = \dfrac{y^2 - x^2}{y^2 + x^2}$ $\quad y(0) = 1$ $\quad$ for $x = 0.2$.

$x_1 = 0.2$, $\quad x_0 = 0$, $\quad h = 0.2$.

$y_1 = y_0 + K$. $\quad$, $y_0 = 1$

$$K_1 = h f(x_0, y_0)$$

$$K_2 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$K_3 = h f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$K_4 = h f(x_0 + h,\ y_0 + K_3)$$

$$K_1 = (0.2)\, f(0, 1) \Rightarrow (0.2)\left[\frac{1^2 - 0}{1^2 + 0}\right] = 0.2$$

$$K_2 = (0.2)\, f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right) = 0.2\, f(0.1, 1.1)$$

$$= 0.2\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right] = 0.196721.$$

$$K_3 = (0.2) f\left[0 + \frac{0.2}{2},\ 1 + \frac{0.196721}{2}\right] = (0.2) f\left[0.1,\right.$$

Solve $(x + y)y' = 1$ $\quad y(0.4) = 1$ $\quad$ at $x = 0.5$.

UNIT - 5

Calculus of Variations and Z- transforms.

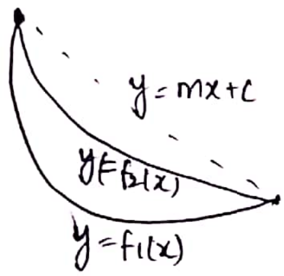
Q) What is the shortest path b/w 2 points.

a) on a plane → straight line

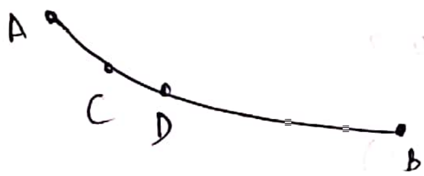
b) on a cylinder → helix

c) on a sphere → circle.

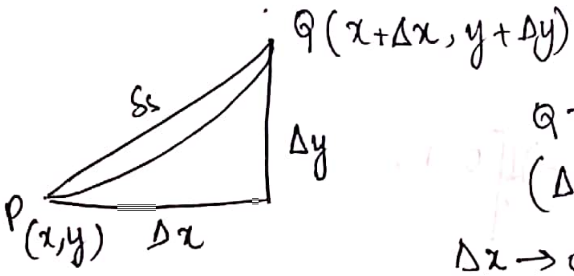
d) Freely hanging cable → $y = c\cosh\left(\frac{x}{c}\right)$ → catenary



Curve along which the particle slides freely without friction in the least time (Brachistochrone problem) (for same particle)



Tautochrone : Irrespective of the position of the object (say A, C, D) they take the same amount of time to reach B.



$Q \to P \qquad \delta s \to \Delta s$

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$\Delta x \to 0 \qquad ds^2 = dx^2 + dy^2$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\int ds = \int \sqrt{1 + y_1^2}\, dx$$

$$S = \int_a^b \sqrt{1 + y_1^2}\, dx$$

Definition :

$$I(y) = \int_a^b f(x, y, y')dx \longrightarrow \text{Functionals}$$

$$f(x, y+\delta y, y'+\delta y') = f(x,y,y') + \delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} + \cdots$$

$$\delta f = \delta y' \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} \rightarrow 1^{st} \text{ variation of the func } f$$

Variation of a functional

$$\delta I = \delta y \frac{dI}{\partial y} + \delta y' \frac{\partial I}{\partial y'}$$

$$= \delta y \frac{\partial}{\partial y} \int_a^b f(x,y,y')dx + \delta y' \frac{\partial}{\partial y'} \int_a^b f(x,y,y')dx$$

$$= \delta y \int_a^b \frac{\partial f}{\partial y} dx + \delta y' \int_a^b \frac{\partial f}{\partial y'} dx$$

$$= \int_a^b \left[\delta y \frac{\partial f}{\partial y} + \delta y' \frac{\partial f}{\partial y'} \right] dx = \int_a^b \delta f dx$$

Variational problem :

The task of finding $y = f(x)$ such that the functional $I = \int_a^b f(x,y,y')dx$ is an extremum

Necessary condition for $I = \int_a^b f(x,y,y')dx$ to be extremum is

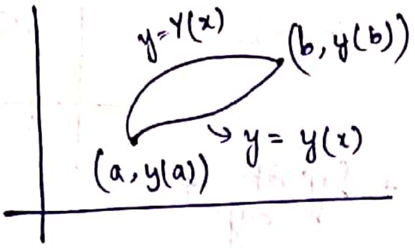
i) first derivative = 0

ii) [Euler's eqn] $= \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

Proof : Consider a curve $y = y(x)$ along which

$$I = \int_a^b f(x,y,y')dx \quad \text{(1)} \quad \text{is extremum}$$

Let $y = Y(x)$ be a neighbouring curve of $y = y(x)$ b/w $(a, y(a))$ & $(b, y(b))$



Now, $Y(x) = y(x) + \varepsilon\eta(x)$

$\eta(a) = 0 \qquad \eta(b) = 0$

$$I = \int_a^b f(x, Y(x), Y'(x))\,dx \rightarrow (2)$$

(2) can be extremum.

$$(2) \Rightarrow I(\varepsilon) = \int_a^b \cancel{f(x)}\, f(x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x))\,dx \rightarrow (3)$$

(3) is extremum at $\delta = 0$ and cond^n is $\frac{dI}{d\varepsilon} = 0$ at $\varepsilon = 0$

$$\frac{dI}{d\varepsilon} = \int_a^b \frac{\partial f}{\partial \varepsilon}(x, y + \varepsilon\eta, y' + \varepsilon\eta')\,dx$$

$$\frac{\partial f}{\partial \varepsilon} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial \varepsilon} + \frac{\partial f}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}$$

$x \rightarrow$ independent variable $\Rightarrow \frac{\partial x}{\partial \varepsilon} = 0$

$$\frac{\partial Y}{\partial \varepsilon} = \eta(x) \qquad \frac{\partial Y'}{\partial \varepsilon} = \eta'(x)$$

$$\frac{dI}{d\varepsilon} = \int_a^b \left[\frac{\partial f}{\partial Y}\eta(x) + \frac{\partial f}{\partial Y'}\eta'(x)\right]dx$$

$$= \int_a^b \frac{\partial f}{\partial Y}\eta(x)\,dx + \int_a^b \frac{\partial f}{\partial Y'}\eta'(x)\,dx$$

Apply integration by parts for 2nd integral.

$$\frac{dI}{d\varepsilon} = \int_a^b \frac{\partial f}{\partial Y}\eta(x)\,dx + \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x=a}^{b} - \int_a^b \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx$$

$$= \int_a^b \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\}\eta(x)\,dx + \left.\frac{\partial f}{\partial y'}\right|_{x=b}\eta(b) - \left.\frac{\partial f}{\partial y'}\right|_{x=a}\eta(a)$$

But $\eta(b) = \eta(a) = 0$

$$\frac{dI}{d\varepsilon} = \int_a^b \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} \eta(x)\, dx$$

at $\varepsilon = 0$ $\quad \frac{dI}{d\varepsilon} = \int_a^b \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} \eta(x)\, dx$

Now $\frac{dI}{d\varepsilon} = 0$ at $\varepsilon = 0$ $\rightarrow \int_a^b \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} \eta(x)\, dx = 0$

for any $\eta(x)$.

0 The result is only possible if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left\{ \frac{\partial f}{\partial y'} \right\} = 0 \longrightarrow \text{Euler's eq}^n$$

14 NOV Alternate forms of Euler's eq^n.

1) $\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'} \right] - \frac{\partial f}{\partial x} = 0$

2) $\frac{\partial f}{\partial y} - y\frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial (y')^2} = 0$

Special cases:

1) f is independent of x.

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

2) f is independent of y

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = \text{constant.}$$

3) f is independent of y'

$$\frac{\partial f}{\partial y} = 0$$

4) f is independent of both x & y.

$$y''\frac{\partial^2 f}{\partial (y')^2} = 0$$

If $\frac{\partial^2 f}{\partial (y')^2} \neq 0 \Rightarrow y'' = 0$

$y' = A \Rightarrow y = Ax + B$.

1) Find the extremal of

$$\int_0^1 \{(y')^2 + 12xy\}\,dx \quad \text{with} \quad y(0)=0,\ y(1)=1$$

$$I = \int_{x_1}^{x_2} f(x,y,y')\,dx$$

$f(x,y,y') = (y')^2 + 12xy$

Euler's eqn : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 12x - \frac{d}{dx}(2y') = 0 \Rightarrow \frac{d}{dx}(y') = 6x$$

$$\Rightarrow y' = \frac{6x^2}{2} + A \Rightarrow \frac{dy}{dx} = \frac{6x^2}{2} + A$$

$$\Rightarrow y = \frac{\cancel{3}x^3}{\cancel{3}} + Ax + B = x^3 + Ax + B.$$

$0 = 0 + B \Rightarrow B = 0$

$1 = 1 + A + B \Rightarrow A = 0$

$\therefore y = x^3$

2) Solve the variational problem

$$\delta\left\{\int_0^{\pi/2} \{(y')^2 - y^2 + 2xy\}\,dx\right\} = 0 \qquad y(0) = 0 = y\left(\frac{\pi}{2}\right)$$

Variational problem is finding y such that $\delta I = 0$.

$$I = \int_0^{\pi/2} ((y')^2 - y^2 + 2xy)\,dx = \int_0^{\pi/2} f(x,y,y')\,dx$$

Euler's eqn : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$(-2y + 2x) - \frac{d}{dx}(2y') = 0$$

$$y'' + y = x \Rightarrow (D^2+1)y = x, \quad D = \frac{d}{dx}$$

$F(D)y = g(x)$, $F(D) = D^2 + 1$

$y = CF + PI$

$AE : F(m) = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = i, -i$

$CF = \{ C_1 \cos(\beta x) + C_2 \sin(\beta x) \} e^{\alpha x}$, $m = \alpha \pm i\beta$

$\alpha = 0$, $\beta = 1$ [i.e $m = \pm i$]

$CF = C_1 \cos x + C_2 \sin(x)$

$PI = \frac{1}{F(D)} g(x) = \frac{1}{D^2+1} x = (1+D^2)^{-1} x$

$= (1 - D^2 + (D^2)^2 + \cdots) x = x$ [ie $D = \frac{d}{dx} \Rightarrow D^2 x = 0$
$D^4 x = 0$, $D^6 x = 0$]

$y = C_1 \cos(x) + C_2 \sin x + x$.

$x = 0 \Rightarrow y = 0 \Rightarrow 0 = C_1$.

$y = C_2 \sin x + x$

$y(\frac{\pi}{2}) = 0 \Rightarrow 0 = C_2 + \frac{\pi}{2} \Rightarrow C_2 = -\frac{\pi}{2}$

$y = -\frac{\pi}{2} \sin x + x$.

1st page straight line concept continued.

$I = \sqrt{1 + (y')^2}\, dx$.

f is independent of both x and y.

$\Rightarrow y'' \frac{\partial^2 f}{\partial (y')^2} = 0$

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+(y')^2}} (2y') = \frac{y'}{\{1+(y')^2\}^{1/2}}$$

$$\frac{\partial^2 f}{\partial (y')^2} = \frac{\{1+(y')^2\}^{1/2} - (y')\left(\frac{1}{2}\right)\{1+(y')^2\}^{-1/2}(2y')}{1+(y')^2}$$

$$\frac{\partial^2 f}{\partial (y')^2} = \frac{1+(y')^2 - (y')^2}{\{1+(y')^2\}^{3/2}} = \frac{1}{\{1+(y')^2\}^{3/2}} \neq 0$$

$\therefore y'' = 0 \Rightarrow y' = Ax \Rightarrow y = Ax + B$.

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18th Nov

$y' = \pm \frac{cx}{\sqrt{1-c^2x^2}}$

Find the curve $y = y(x)$ which passes through $(1,0)$ & $(2,1)$, $I = \int_0^1 \frac{1}{x}\sqrt{1+(y')^2}\,dx$ can be extremized.

$$f(x,y,y') = \frac{1}{x}\sqrt{1+(y')^2}$$

$(1+(y')^2)^{1/2}$

Euler's eq^n: $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\frac{d}{dy'} = \frac{1}{2}\left(1+(y')^2\right)^{-1/2}$

$$0 - \cancel{\frac{d}{dx}}\cancel{\left(\frac{y}{x}\right)} \quad \frac{\partial f}{\partial y'} = \text{constant}$$

Independent of $y \Rightarrow \frac{\partial f}{\partial y} = 0$.

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = \text{constant}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{x}\frac{1}{\cancel{2}\sqrt{1+y_1^2}}(\cancel{2}y_1) = \frac{y_1}{x\sqrt{1+y_1^2}}$$

$$\frac{\partial f}{\partial y'} = \text{constant} \Rightarrow \frac{y_1}{x\sqrt{1+y_1^2}} = \text{const.}$$

$$\Rightarrow y_1^2 = x^2(1+y_1^2)\,(\text{constant})$$

let constant $= A^2$

$$y_1^2 = x^2(1+y_1^2)A^2$$

$$y_1^2(1-x^2A^2) = x^2A^2$$

$$y_1^2 = \frac{x^2A^2}{1-x^2A^2} \Rightarrow y_1 = \pm\frac{xA}{\sqrt{1-x^2A^2}}$$

$$\frac{dy}{dx} = \frac{\pm xA}{\sqrt{1-x^2A^2}} \Rightarrow \int dy = \pm\int\frac{xA\,dx}{\sqrt{1-x^2A^2}} + B$$

$$1-x^2A^2 = t \Rightarrow -2xA^2dx = dt$$

$$Ax\,dx = \frac{dt}{-2A}$$

$$y = \pm\int\frac{-dt/2A}{\sqrt{t}} + B = \mp\frac{1}{A}\sqrt{t} + B$$

$$y - B = \mp\frac{1}{A}\sqrt{1-A^2x^2}$$

$$(y-B)^2 = \frac{1}{A^2}(1-A^2x^2) \Rightarrow x^2+(y-B)^2 = \frac{1}{A^2} \rightarrow \text{Circle}$$

Curve passes through $(1,0)$, $(2,1)$

$$1+(0-B)^2 = \frac{1}{A^2} \Rightarrow 1+B^2 = \frac{1}{A^2}$$

$$4+(1-B)^2 = \frac{1}{A^2} \Rightarrow 4+1+B^2-2B = \frac{1}{A^2}$$

$$\Rightarrow -4+2B = 0 \Rightarrow B = 2$$

$$\Rightarrow \frac{1}{A^2} = 5$$

$$\Rightarrow x^2+(y-2)^2 = 5$$

Q) $\delta \int_0^{\pi/2} [(y')^2 - y^2 + 2xy]\,dx = 0$, $y(0) = y(\frac{\pi}{2}) = 0$

Q) Find the extremal of $\int_{x_1}^{x_2} [x^2(y')^2 + 2y^2 + 2xy]\,dx$

$$f(x,y,y') = x^2(y')^2 + 2y^2 + 2xy$$

Euler's eq^n : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 4y+2x - \frac{d}{dx}\left(x^2 2y'(1)\right) = 0$$

$$\Rightarrow \cancel{4y+2x-2xy'}$$

$$4y+2x - \left[2y'(2x) - 2x^2y''\right] = 0$$

$$x^2y'' + 2xy' - 2y = x \rightarrow \text{Cauchy's DE}$$

$$x = e^z \Rightarrow xD = D_1 \qquad D = \frac{d}{dx} \qquad D_1 = \frac{d}{dz}$$

$$x^2D^2 = D_1(D_1-1)$$

$$\left[x^2D^2 + 2xD - 2\right]y = x$$

$$\left[D_1(D_1-1) + 2D_1 - 2\right]y = e^z$$

$$\cancel{\left[D_1 - 3D_1 - 2\right]y = e^z}$$

$$\cancel{F(D_1) = D_1^2 - 3D_1 - 2}$$

$$\cancel{AE : F(m) = 0 \Rightarrow m^2 - 3m - 2 = 0}$$

$$\cancel{m = \frac{3 \pm \sqrt{17}}{2} = \frac{m+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}}$$

$$\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = \frac{d}{dz}\left(\frac{1}{x}\right)$$

$$\text{|||}^{rly} \quad x^2D^2 = D_1(D_1-1)$$

$m_1 = \frac{3+\sqrt{17}}{2}$ $\quad m_2 = \frac{3-\sqrt{17}}{2}$

$\cancel{CF = c_1 e^{m_1 z} + c_2 e^{m_2 z}}$

$\cancel{CF = c_1 x^{m_1} + c_2}$

$[D_1^2 + 1D_1 - 2]y = e^z$

$F(D_1) = D_1^2 + D_1 - 2$

AE : $F(m) = 0 \Rightarrow m^2 + m - 2 = 0$

$m = \frac{-1 \pm \sqrt{1+8}}{2} = 1, -2.$

$CF = c_1 e^{m_1 z} + c_2 e^{m_2 z} = c_1 e^z + c_2 e^{-2z}$

$CF = c_1 x + c_2 x^{-2}$

$PI = \frac{1}{F(D_1)} e^z \Rightarrow z\left[\frac{1}{F'(D_1)} e^z\right] = z\left[\frac{1}{2+1} e^z\right]$

$= \frac{z}{3} e^z = \frac{(\log x)x}{3}$

$y = CF + PI = c_1 x + c_2 x^{-2} + \frac{x}{3}\log x$

$\left[\because PI = \frac{1}{F(a)} e^{az}\right.$ if $F(a) \neq 0$

$F'(D_1) = 2D_1 + 1$

$PI = z\left[\frac{1}{F'(a)} e^{az}\right]$

if $F(a) = 0$ & $F'(a) \neq 0$.

Q) Find the extremal of

$\int \{(y')^2 - y^2 + 2y\sec x\} dx$

$f(x, y, y') = (y')^2 - y^2 + 2y\sec x.$

Euler's eq^n : $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$= -2y + 2\sec x - \frac{d}{dx}(2y') = 0$

$\Rightarrow y'' + y = \sec x$

$\Rightarrow (D^2+1)y = \sec x$

$F(D) = D^2 + 1$

AE : $m^2 + 1 = 0 \Rightarrow m = \pm i$

$CF = c_1\cos x + c_2 \sin x$

let $y = Au + Bv$ be soln of $F(D)y = R$.

$A = -\int \frac{vR}{W}$ $\quad B = \int \frac{uR}{W}$ $\quad u = \cos x \quad v = \sin x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$$

$$A = \int \frac{(-\sin x)\sec x}{1} dx = -\int \tan x \, dx = -\log(\sec x) = \log(\cos x)$$

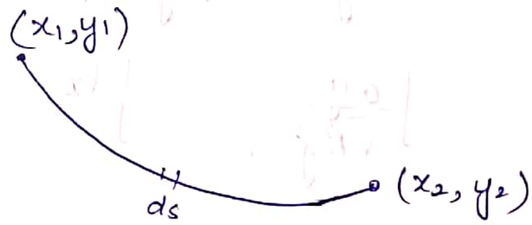
$$B = \int \frac{(\cos x)\sec x}{1} dx = \int dx = x$$

$$PI = Au + Bv = (\cos x)\log(\cos x) + x\sin(x)$$

$$y = c_1\cos x + c_2\sin x + (\cos x)\log(\cos x) + x\sin(x)$$

Q) Consider a cable hanging freely btw the points (x_1, y_1) & (x_2, y_2). Show that the shape of the curve formed by the hanging cable is a catenary.

[HANGING CABLE PROBLEM]



Let ρ be linear density of the cable.

Potential Energy PE of the element of the cable of length ds which is at a height y. $PE = mgy = (\rho\, ds)gy$

$\because$ linear density $\rho = \frac{\text{mass}}{\text{length}}$

$$(ds)^2 = (dx)^2 + (dy)^2 \Rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$ds = \sqrt{1 + y_1^2}\, dx$$

$$PE = (\rho\sqrt{1+y_1^2}\, dx)gy$$

$$\text{Total potential Energy} = \int_{x_1}^{x_2} \rho g y \sqrt{1+y_1^2}\, dx$$

$$I = \int_{x_1}^{x} \rho g y \sqrt{1+y_1^2}\, dx.$$

Task: To find $y = y(x)$ such that I is extremized.

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx.$$

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$$f(x, y, y') = \rho g y \sqrt{1+y_1^2}$$

f is independent of x

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}.$$

$$\rho g \left[y\sqrt{1+y_1^2} - y' y \frac{y_1}{\sqrt{1+y_1^2}} \right] = \text{const}.$$

$$\rho g \left[\frac{y(1+y_1^2 - y_1^2)}{\sqrt{1+y_1^2}} \right] = \text{constant}.$$

$$\frac{\rho g y}{\sqrt{1+y_1^2}} = \text{constant} \Rightarrow \quad \frac{\sqrt{1+y_1^2}}{y} = \text{constant} = A.$$

$$1 + y_1^2 = A^2 y^2$$

$$y_1^2 = A^2 y^2 - 1$$

$$y_1 = \sqrt{A^2 y^2 - 1}$$

$$\int \frac{dy}{\sqrt{A^2y^2 - 1}} = \int dx \Rightarrow \quad \cos^2\theta + \sin^2\theta = 1$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\Rightarrow \cosh^2\theta - 1 = \sinh^2\theta$$

$$Ay = \cosh(\theta) \Rightarrow A\,dy = \sinh(\theta)d\theta$$

$$\int \frac{\frac{1}{A}\sinh\theta d\theta}{\sqrt{\cosh^2\theta - 1}} = \int dx$$

$$\frac{1}{A}\int \frac{\sinh\theta\, d\theta}{\sin\theta} = x + B$$

$$\frac{1}{A}\int d\theta = x + B \Rightarrow \quad \frac{1}{A}(\theta) = x + B$$

$$\frac{1}{A}\cosh^{-1}(Ay) = x + B \Rightarrow \cosh^{-1}(Ay) = Ax + AB$$

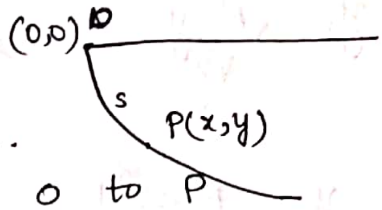
$$Ay = \cosh(Ax + AB)$$

$$\frac{1}{A} = c \Rightarrow \quad y = c\cosh\left(\frac{x+B}{c}\right)$$

Q. [BRACHISTOCHRONE PROBLEM]

Brachistos + chrones
shortest or least + time

Consider a particle at origin sliding freely without friction along a curve $y = y(x)$.



Let P be any point on the curve.

~~PE @ R~~ The work done in moving O to P is equal to KE at P.

$$[\text{PE at } P - \text{PE at } O] = \text{KE at } P$$

$$mgy - mg(0) = \frac{1}{2}mv^2$$

$$v = \frac{ds}{dt}$$

$$\cancel{m}gy = \frac{1}{2}\cancel{m}\left(\frac{ds}{dt}\right)^2 \Rightarrow 2gy = \left(\frac{ds}{dt}\right)^2$$

$$dt^2 = \frac{(ds)^2}{2gy} \Rightarrow \int_0^T dt = \int_0^{x_1} \frac{ds}{\sqrt{2gy}}$$

$$T = \int_0^{x_1} \frac{ds}{\sqrt{2gy}} \qquad (ds)^2 = (dx)^2 + (dy)^2 = (dx)^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

$$ds = \sqrt{1+y_1^2}\,dx$$

$$T = \int_0^{x_1} \frac{\sqrt{1+y_1^2}\,dx}{\sqrt{2gy}}$$

Task : find $y = y(x)$ such that T is minimum.

$$f(x, y, y') = \frac{\sqrt{1+y_1^2}}{\sqrt{2gy}}$$

f is independent of x

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

$$\frac{\sqrt{1+y_1^2}}{\sqrt{2gy}} - \frac{y'}{\sqrt{2gy}}\left(\frac{y_1}{\sqrt{1+y_1^2}}\right) \Rightarrow \frac{1}{\sqrt{2gy}}\left[\frac{1+\cancel{y_1^2} - \cancel{y_1^2}}{\sqrt{1+y_1^2}}\right] = \text{const}$$

$$\sqrt{y}\sqrt{1+y_1^2} = \text{constant} = A$$

$$y(1+y_1^2) = A^2$$

$$1+y_1^2 = \frac{A^2}{y} \Rightarrow y_1^2 = \frac{A^2}{y} - 1$$

$$y_1 = \frac{\sqrt{A^2-y}}{\sqrt{y}} \Rightarrow \frac{\sqrt{y}}{\sqrt{A^2-y}}\,dy = dx$$

$$y = A^2\sin^2\theta \qquad dy = A^2(2\sin\theta)(\cos\theta)\,d\theta$$

$$\frac{\sqrt{A^2\sin^2\theta}}{\sqrt{A^2 - A^2\sin^2\theta}}\,A^2(2\sin\theta)(\cos\theta)\,d\theta = \int dx$$

$$= 2A^2\int \frac{\sin^2\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}} = \int dx$$

$$= 2A^2\int \sin^2\theta\,d\theta = dx \Rightarrow A^2\int(1-\cos 2\theta)\,d\theta = \int dx$$

$$= A^2\left\{\theta - \frac{\sin(2\theta)}{2}\right\} + \frac{c}{2} = x$$

$$= \quad x = \frac{A^2}{2}\left(2\theta - \sin(2\theta)\right) + c$$

$$x = a(t-\sin t) + c$$

$$y = \frac{A^2(1-\cos 2\theta)}{2} = a(1-\cos t)$$

at $x=0,\ y=0$

$\therefore\ y=0 \Rightarrow t=0$

$\Rightarrow\ 0 = a(0-0)+c \Rightarrow c=0$

$\therefore\ x = a(t-\sin t)$ and $y = a(1-\cos t) \rightarrow$ cycloid. - path travelled

Z transforms.

If $u_n = f(n) \quad n = 0, 1, 2, 3, \ldots$

& $u_n = 0 \quad$ for $n<0$.

$Z[u_n] = U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

Inverse z- transform $\quad Z^{-1}[U(z)] = u_n$

$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots \quad \left|\frac{a}{z}\right|<1$

$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a} \quad |z|>|a|$

$\left[\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r|<1\right]$

$Z[1] = \frac{z}{z-1}$

Prove that :

$Z[n^k] = -z\frac{d}{dz} Z[n^{k-1}]$

Consider $\quad Z[n^{k-1}] = \sum_{n=0}^{\infty} n^{k-1} z^{-n}$

$\frac{d}{dz}\left(Z[n^{k-1}]\right) = \sum_{n=0}^{\infty} n^{k-1}(-n) z^{-n-1}$

$= -\sum_{n=0}^{\infty} n^k z^{-n-1} = -\sum_{n=0}^{\infty} n^k \frac{z^{-n}}{z}$

$= -\frac{1}{z}\left[\sum_{n=0}^{\infty}(n^k z^{-n})\right] = -\frac{1}{z} Z[n^k]$

$Z[n^k] = -z\frac{d}{dz}\left[Z(n^{k-1})\right]$

$k=1 \Rightarrow Z[n] = -z\frac{d}{dz} Z[1]$

$Z[n] = -z\frac{d}{dz}\left\{\frac{z}{z-1}\right\} = -z\left\{\frac{(z-1)(1) - z(1)}{(z-1)^2}\right\}$

$= -z\left\{\frac{-1}{(z-1)^2}\right\} = \frac{z}{(z-1)^2}$

$Z[n^2] = -z\frac{d}{dz} Z[n] = -z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\} = \frac{z(z+1)}{(z-1)^3}$

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$z[n^3] =$

Properties of z-transform

1) Linearity property

$$Z[aU_n + bV_n - cW_n] = aU(z) + bV(z) - cW(z)$$

$$Z[aU_n + bV_n - CW_n] = \sum_{n=0}^{\infty} (aU_n + bV_n - cW_n) z^{-n}$$

$$= a\sum_{n=0}^{\infty} U_n z^{-n} + b\sum_{n=0}^{\infty} V_n z^{-n} - c\sum_{n=0}^{\infty} W_n z^{-n}$$

$$= aZ[U_n] + bZ[V_n] - cZ[W_n]$$

2) Damping rule.

$$\left[Z[U_n] = \sum_{n=0}^{\infty} U_n z^{-n} \right. \quad (1)$$

$$Z[a^{-n}U_n] = \sum_{n=0}^{\infty} a^{-n} U_n z^{-n}$$

$$Z[a^{-n}U_n] = \sum_{n=0}^{\infty} U_n (az)^{-n}$$

From (1)

$$Z[a^{-n}U_n] = U(az)$$

$$Z[a^n U_n] = U\left(\frac{z}{a}\right)$$

3) Shifting Property

a) Right Shifting Property

$$Z[U_{n-k}] = z^{-k} U(z)$$

Now, $U_{n-k} = 0$ for $n < k$

$$Z[U_{n-k}] = \sum_{n=0}^{\infty} U_{n-k} z^{-n} \quad \therefore \quad \sum_{n=0}^{k-1} U_{n-k} z^{-n} + \sum_{n=k}^{\infty} U_{n-k} z^{-n}$$

$$= 0 + \sum_{n=k}^{\infty} U_{n-k} z^{-n}$$

$n - k = r$

$n = k \Rightarrow r = 0.$

$n \to \infty \Rightarrow r = \infty$

$$= Z[U_{n-k}] = \sum_{r=0}^{\infty} U_r z^{-(r+k)}$$

$$= \sum_{r=0}^{\infty} U_r z^{-r} z^{-k} = z^{-k} \sum_{r=0}^{\infty} U_r z^{-r} z^{-k} = z^{-k} \sum_{n=0}^{\infty} U_r z^{-r}$$

$$= z^{-k} Z[U_n] = z^{-k} U(z)$$

b) Left Shifting property

$$Z[U_{n+k}] = z^k \left[U(z) - U_0 - U_1 z^{-1} - U_2 z^{-2} \dots - U_{k-1} z^{-(k-1)} \right]$$

$$Z[U_{n+k}] = \sum_{n=0}^{\infty} U_{n+k} z^{-n}$$

$n + k = r \Rightarrow n = r - k.$

$n = 0 \Rightarrow r = k$

$n \to \infty \Rightarrow r \to \infty$

-① $$Z[U_{n+k}] = \sum_{r=k}^{\infty} U_r z^{-r-k} = \sum_{r=k}$$

$$= \sum_{r=0}^{k-1} U_r z^{-(r-k)} - \sum_{r=0}^{k-1} U_r z^{-(r-k)} + \sum_{r=k}^{\infty} U_r z^{-(r-k)}$$

$$= \sum_{r=0}^{k-1} U_r z^{-(r-k)} + \sum_{r=k}^{\infty} U_r z^{-(r-k)} - \sum_{r=0}^{k} U_r z^{-(r-k)}$$

$$= \sum_{r=0}^{\infty} U_r z^{-(r-k)} - \sum_{r=0}^{k-1} U_r z^{-(r-k)}$$

$$Z[U_{n+k}] = \sum_{n=0}^{\infty} U_r z^{-r} z^k - \sum_{r=0}^{k-1} U_r z^{-r} z^k$$

$$= z^k \left[\sum_{r=0}^{\infty} U_r z^{-r} - \sum_{r=0}^{k-1} U_r z^{-r} \right]$$

$$= z^k \left[Z[U_n] - U_0 - U_1 z^{-1} - U_2 z^{-2} \dots - U_{k-1} z^{-(k-1)} \right]$$

$$= z^k \left[U(z) - U_0 - U_1 z^{-1} - U_2 z^{-2} \dots - U_{k-1} z^{-1(k-1)} \right]$$

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$$\Rightarrow Z[U_{n+1}] = z[U(z) - U_0]$$

$$Z[U_{n+2}] = z^2[U(z) - U_0 - U_1 z^{-1}]$$

27th Nov

Final value Theorem

If $Z[U_n] = U(z)$

then, $\lim_{z\to 1}[(z-1)U(z)] = \lim_{n\to\infty} U_n$

$$Z[U_n] = U(z)$$

$$Z[U_{n+1}] = z[U(z) - U_0]$$

$$Z[U_{n+1}] - Z[U_n] = (z-1)U(z) - zU_0$$

$$\sum_{n=0}^{\infty} U_{n+1} z^{-n} - \sum_{n=0}^{\infty} U_n z^{-n} = (z-1)U(z) - zU_0$$

$$\sum_{n=0}^{\infty} (U_{n+1} - U_n) z^{-n} = (z-1)U(z) - zU_0$$

$$\lim_{z\to 1} \sum_{n=0}^{\infty} (U_{n+1} - U_n) z^{-n} = \lim_{z\to 1} \{(z-1)U(z)\} - \lim_{z\to 1} (z$$

$$\sum_{n=0}^{\infty} (U_{n+1} - U_n) = \lim_{z\to 1} (z-1)U(z) - U_0$$

$$(U_1 - U_0) + (U_2 - U_1) + (U_3 - U_2) + \dots = \lim_{z\to 1} (z-1)U(z) -$$

$$-U_0 + \lim_{n\to\infty} U_n = \lim_{z\to 1} (z-1)U(z) - U_0$$

$$\lim_{n\to\infty} U_n = \lim_{z\to 1} (z-1)U(z)$$

9) $Z[U_n] = \dfrac{5z^2+3z+12}{(z-1)^4} = U(z)$ find U_2 and U_3

Ans

$$\lim_{z\to\infty} U(z) = U_0$$

$$\lim_{z\to\infty} Z[U_{n+k}] = U_k = \lim_{z\to\infty} z^k\left[U(z) - U_0 - U_1 z^{-1} - U_2 z^{-2} \cdots - U_{k-1} z^{-(k-1)}\right]$$

$$U_0 = \lim_{z\to\infty} \frac{5z^2+3z+12}{(z-1)^4} = \lim_{z\to\infty} \frac{\cancel{z^2}\left(5+\frac{3}{z}+\frac{12}{z^2}\right)}{\cancel{z^4}^{2}\left(1-\frac{1}{z}\right)^4} = 0$$

$$U_1 = \lim_{z\to\infty} Z[U_{n+1}] = \lim_{z\to\infty} z\left[U(z)-U_0\right]$$

$$= \lim_{z\to\infty} \frac{z(5z^2+3z+12)}{(z-1)^4} \Rightarrow \frac{\cancel{z^3}\left(5+\frac{3}{z}+\frac{12}{z^2}\right)}{z\cancel{z^4}\left(1-\frac{1}{z}\right)^4}$$

$$U_1 = 0$$

$$U_2 = \lim_{z\to\infty} Z[U_{n+2}]$$

$$= \lim_{z\to\infty} z^2\left[U(z) - U_0 - U_1 z^{-1}\right]$$

$$= \lim_{z\to\infty} \frac{z^2\left[5z^2+3z+12\right]}{(z-1)^4} = \frac{\cancel{z^4}\left[5+\frac{3}{z}+\frac{12}{z^2}\right]}{\cancel{z^4}\left(1-\frac{1}{z}\right)^4} = 5$$

$$U_3 = \lim_{z\to\infty} Z[U_{n+3}]$$

$$= \lim_{z\to\infty} z^3\left[U(z) - U_0 - U_1 z^{-1} - U_2 z^{-2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{5z^2+3z+12}{(z-1)^4} - 0 - 0 - 5z^{-2}\right]$$

$$\Rightarrow z^3\left[\frac{(5z^2+3z+12)z^2 - 5(z-1)^4}{z^2(z-1)^4}\right]$$

$$= \lim_{z\to\infty} \frac{z^3\left\{5z^4+3z^3+12z^2 - 5\left(z^4-4z^3+6z^2-4z+1\right)\right\}}{z^6\left(1-\frac{1}{z}\right)^4}$$

$$\lim_{z\to\infty} \frac{z^3\left[23z^3-18z^2+20z-5\right]}{z^6\left(1-\frac{1}{z}\right)^4}$$

$$\lim_{z\to\infty} = \frac{\cancel{z^6}\left[23-\frac{18}{z}+\frac{20}{z^2}-\frac{5}{z^3}\right]}{\cancel{z^6}\left(1-\frac{1}{z}\right)^4} = 23.$$

Q) $$Z[U_n] = \frac{2z^2+3z+1}{(z-1)^3} = U(z)$$

find U_1, U_2 and U_3

$$\lim_{z\to\infty} U(z) = U_0$$

$$\lim_{z\to\infty} Z[U_{n+k}] = U_k = \lim_{z\to\infty} z^k\left[U(z) - U_0 - U_1 z^{-1} - U_2 z^{-2} \dots - U_{k-1} z^{-(k-1)}\right]$$

$$U_0 = \lim_{z\to\infty} \frac{2z^2+3z+1}{(z-1)^3} = \frac{\cancel{z^2}\left(2+\frac{3}{z}+1\right)}{z\,\cancel{z^3}\left(\cancel{1}-\frac{1}{z}\right)^3} = 0$$

$$U_1 = \lim_{z\to\infty} Z[U_{n+1}] = U_1 = \lim_{z\to\infty} z\left[U(z) - U_0\right]$$

$$\lim_{z\to\infty} z\left[\frac{2z^2+3z+1}{(z-1)^3}\right] = \frac{\cancel{z^3}\left[2+\frac{3\cancel{z}}{z}+\frac{1}{z^2}\right]}{\cancel{z^3}\left(1-\frac{1}{z}\right)^3}$$

$$U_1 = \cancel{z}.2$$

$$U_2 = \lim_{z\to\infty} Z[U_{n+2}] = U_2 = z^2\left[U(z) - U_0 - U_1 z^{-1}\right]$$

$$= z^2\left[\frac{2z^2+3z+1}{(z-1)^3} - \frac{\cancel{x}}{z}\right]$$

$$= z^2\left[\frac{(2z^2+3z+1)z - 2(z-1)^3}{z(z-1)^3}\right]$$

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$$= z^2\left[\frac{2z^3+3z^2+z-2(z^3-3z^2+3z-1)}{z(z-1)^3}\right]$$

$$= z^2\left[\frac{9z^2-5z+2}{z(z-1)^3}\right] = \frac{\cancel{z^4}\left[9-\frac{5}{z}+\frac{2}{z^2}\right]}{\cancel{z^4}\left(1-\frac{1}{z}\right)^3}$$

$$U_2 = 9$$

$$U_3 = \lim_{z\to\infty} Z[U_{n+3}] = \lim_{z\to\infty} z^3\left[U(z)-U_0-U_1z^{-1}-U_2z^{-2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+1}{(z-1)^3}-0-\frac{2}{z}-\frac{9}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{z^2(2z^2+3z+1)-2(z-1)^3z-9(z-1)^3}{z^2(z-1)^3}\right]$$

$$= \lim_{z\to\infty} \cancel{z^3}z\left[\frac{2z^4+3z^3+z^2-2z^4+6z^3-6z^2+2z-9z^3+27z^2-27z+9}{\cancel{z^2}(z-1)^3}\right]$$

$$= \lim_{z\to\infty} z\left[\frac{(2-2)z^4+(3+6-9)z^3+(1-6+27)z^2+(2-27)z+9}{(z-1)^3}\right]$$

$$= \lim_{z\to\infty} \frac{z(22z^2-25z+9)}{(z-1)^3}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^3}\left(22-\frac{25}{z}+\frac{9}{z^2}\right)}{(\cancel{z^3})\left(1-\frac{1}{z}\right)^3} = 22.$$

Q) $U(z) = \dfrac{2z^2+5z+4}{(z-1)^4}$ find U_2, U_3

Z transforms

U_n	$Z[U_n]$
1	$\frac{z}{z-1}$
a^n	$\frac{z}{z-a}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z(z+1)}{(z-1)^3}$
$\sin(n\theta)$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
$\cos(n\theta)$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
$\sinh(n\theta)$	$\frac{z\sinh\theta}{z^2-2z\cosh\theta+1}$
$Z[n^{k+1}]$	$-z\frac{d}{dz}\{Z[n^k]\}$
$Z[nU_n]$	$-z\frac{d}{dz}\{Z[U_n]\}$
$Z[a^nU]$	$U(z/a)$
$Z[U_{n+k}]$	$z^k[U(z)-U_0-U_1z^{-1}-U_2z^{-2}-\dots-U_{(k-1)}z^{-(k-1)}$
$Z[U_{n-k}]$	$z^{-k}U(z)$.
$\cosh(n\theta)$	$\frac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}$

Find the inverse z transform of $\frac{z}{(z-a)^2}$

$$Z^{-1}[U(z)] = U_n$$

$$U(z) = \frac{z}{(z-a)^2}$$

w.k.t

$$Z[n] = \frac{z}{(z-1)^2}$$

Now $$U(z) = \frac{z}{(z-a)^2} = \frac{z}{a^2\left(\frac{z}{a}-1\right)^2} = \frac{1}{a}\frac{z/a}{\left(\frac{z}{a}-1\right)^2}$$

$$U\left(\frac{z}{a}\right) = \frac{z/a}{(z/a-1)^2}$$

$$Z[a^nU_n] = U(z/a)$$

$$z^{-1}\left[\frac{z}{(z-a)^2}\right] = \frac{1}{a} z^{-1}\left[U(z/a)\right] = \frac{1}{a} a^n U_n$$

$$U_n = z^{-1}\left[U(z)\right] = z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

$$\therefore\ z^{-1}\left[\frac{z}{(z-a)^2}\right] = \frac{1}{a} a^n n = a^{n-1} n.$$

Q) Find the inverse z transform of $\dfrac{2z^2+3z}{(z+2)(z-4)}$

$$\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}.$$

when $z=-2 \Rightarrow -1 = A(-6) = A = \frac{1}{6}$

$z = 4 \Rightarrow 11 = B(6) \Rightarrow B = 11/6.$

$$2z+3 = A(z-4) + B(z+2).$$

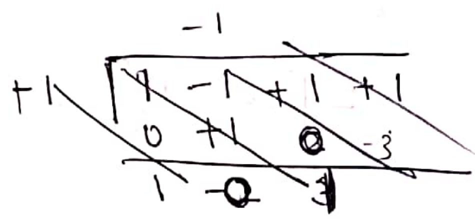
$$\frac{U(z)}{z} = \frac{\frac{1}{6}}{z+2} + \frac{11/6}{z-4}$$

$$U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

$$z^{-1}\left[U(z)\right] = \frac{1}{6} z^{-1}\left[\frac{z}{z-(-2)}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$U_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

Q) $\dfrac{z}{z^3 - z^2 + z + 1}$



$$\frac{U(z)}{z} = \frac{1}{z^3 - z^2 + z + 1}$$

$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} \dots + a_n = 0$

$r_1, r_2, r_3 \dots\dots r_n$ are roots.

then $f(x) = (x-r_1)(x-r_2)(x-r_3)\dots(x-r_n)$

$$U(z) = \frac{z}{(z-1)(z-i)(z+i)} = \frac{z}{(z-1)(z^2+1)}$$

$$\frac{U(z)}{z} = \frac{1}{(z-1)(z^2+1)} = \frac{A}{(z-1)} + \frac{Bz+C}{(z^2+1)}$$

$$1 = A(z^2+1) + (Bz+C)(z-1)$$

$z=1 \Rightarrow 1 = A(1+1) + 0 \Rightarrow A = \frac{1}{2}$

$z=0 \Rightarrow 1 = A + C(-1) \Rightarrow C = A-1 = \frac{1}{2} - 1 = \frac{-1}{2}$

Coeff of $z^2 = 0 \Rightarrow A+B = 0 \Rightarrow B = -A = -\frac{1}{2}$

$$\frac{U(z)}{z} = \frac{1/2}{(z-1)} + \frac{-\frac{1}{2}z - \frac{1}{2}}{(z^2+1)}$$

$$U(z) = \frac{1}{2}\frac{z}{(z-1)} - \frac{1}{2}\frac{(z+1)z}{(z^2+1)} \Rightarrow \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right.$$

$$Z[\sinh n\theta] = \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1} \quad |||^{rly} \text{ for } [\cosh(n\theta)]$$

$\neq 0$ for any θ.

$\frac{e^{\theta}+e^{-\theta}}{2}$.

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z[\cos(n\theta)] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2+1}$$

$$Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$$

$$Z^{-1}[U(z)] = \frac{1}{2}\left\{Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z^2}{z^2+1}\right] - Z^{-1}\left[\frac{z}{z^2+1}\right]\right\}$$

$$= \frac{1}{2}\left\{1 - \cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right\}$$

$U(z) = \frac{8z - z^3}{(4-z)^3}$

Q) $\frac{z}{(z+3)^2(z-2)}$

$\frac{U(z)}{z} = \frac{1}{(z+3)^2(z-2)} = \frac{A}{(z+3)} + \frac{B}{(z+3)^2} + \frac{C}{(z-2)}$

$1 = A(z+3)(z-2) + B(z-2) + C(z+3)^2$

$z = -3 \Rightarrow 1 = B(-5) \Rightarrow B = 1/-5$

$z = 2 \Rightarrow 1 = C(5)^2 \Rightarrow C = \frac{1}{25}$

Coeff of $z^2 = 0$

$0 = A + C \Rightarrow A = -C = \frac{-1}{25}$

$\frac{U(z)}{z} = \frac{\frac{-1}{25}}{z+3} + \frac{\frac{-1}{5}}{(z+3)^2} + \frac{\frac{1}{25}}{z-2}$

$U(z) = \frac{-1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2} + \frac{1}{25}\frac{z}{z-2}$

$z^{-1}[U(z)] = \frac{-1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{9\left(\frac{z}{-3}-1\right)^2}\right] + \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right]$

$z^{-1}\left[\frac{z}{9\left(\frac{z}{-3}-1\right)^2}\right] \Rightarrow z^{-1}\left[\frac{-1}{3}\frac{z/-3}{\left(\frac{z}{-3}-1\right)^2}\right] = \cancel{\frac{-1}{3} V_1\left(\frac{z}{-3}\right)}$ $\because z[a^n V_n] = V\left(\frac{z}{a}\right)$

$= z^{-1}\left[\frac{-1}{3} V\left(\frac{z}{-3}\right)\right] = \frac{-1}{3}(-3)^n V_n$

$V_n = z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

$z^{-1}[U(z)] = \frac{-1}{25}(-3)^n - \frac{1}{5}\left\{\frac{-1}{3}(-3)^n n\right\} + \frac{1}{25} 2^n$

29th Nov

$$U(z) = \frac{8z - z^3}{(4-z)^3}$$

$$U(z) = \frac{z^3 - 8z}{(z-4)^3}$$

$$\frac{U(z)}{z} = \frac{z^2-8}{(z-4)^3} = \frac{A}{(z-4)} + \frac{B}{(z-4)^2} + \frac{C}{(z-4)^3}$$

$z = 4 \Rightarrow 16 - 8 = A(z-4)^2 + B(z-4) + C$

$C = 8.$

$$Z[n^2] = \frac{z(z+1)}{(z-1)^3}$$

$$Z[4^n n^2] = \frac{\left(\frac{z}{4}\right)\left(\frac{z}{4}+1\right)}{\left(\frac{z}{4}-1\right)^3} = \frac{\frac{z}{16}(z+4)}{\frac{1}{64}(z-4)^3}$$

$[Z[a^n u_n] = $

$$= \frac{4z(z+4)}{(z-4)^3}$$

Similarly: $Z[4^n n] = \frac{z/4}{(z/4-1)^2} = \frac{\frac{1}{4}}{\frac{1}{16}}\left\{\frac{z}{(z-4)^2}\right\} = \frac{4z}{(z-4)^2}$

Let $U(z) = \frac{Az}{z-4} + B\frac{4z}{(z-4)^2} + C\frac{4z(z+4)}{(z-4)^3}$

$$\frac{U(z)}{z} = \frac{z^2-8}{(z-4)^2} = \frac{A}{z-4} + \frac{4B}{(z-4)^2} + \frac{4C(z+4)}{(z-4)^3}$$

$z^2 - 8 = A(z-4)^2 + 4B(z-4) + 4C(z+4)$

$z = 4 \Rightarrow 16 - 8 = 0 + 0 + 4C(8)$

$C = \frac{8}{32} = \frac{1}{4}$

Coeff of $z^2 = 1 \Rightarrow A = 1$

Coeff of $z = 0 \Rightarrow -8A + 4B + 4C = 0$

$4B + 4C = 8A = 8 \Rightarrow B + C = 2$

Const $= -8 \Rightarrow 16A - 16B + 16C = -8$

$16 - 16B + 16C = -8$

$16B - 16C = 24 \Rightarrow 2B - 2C = 3$

$4B = 7 \Rightarrow B = \frac{7}{4} \Rightarrow c = \frac{1}{4}.$

$$\frac{U(z)}{z} = \frac{1}{z-4} + \frac{4\left(\frac{7}{4}\right)}{(z-4)^2} + \frac{4\left(\frac{1}{4}\right)(z+4)}{(z-4)^3}$$

$$U(z) = \frac{z}{(z-4)} + \frac{7}{4}\frac{4z}{(z-4)^2} + \frac{1}{4}\frac{4z(z+4)}{(z-4)^3}$$

HW

① $\left(\frac{z}{z-a}\right)^2$ ② $\left(\frac{z}{z-1}\right)^3$ find the inverse z transform.

Q) $U_{n+2} + 4U_{n+1} + 3U_n = 3^n$ $U_0 = 0, U_1 = 1$

step 1 Apply z transform throughout

$$z[U_{n+2}] + 4z[U_{n+1}] + 3z[U_n] = z[3^n]$$

step 2 : $z^2[U(z) - U_0 - U_1 z^{-1}] + 4z[U(z) - U_0] + 3U(z) = \frac{z}{z-3}$

$$z^2\left[U(z) - 0 - \frac{1}{z}\right] + 4z[U(z) - 0] + 3U(z) = \frac{z}{z-3}$$

$$\{z^2 + 4z + 3\}U(z) - z = \frac{z}{z-3}$$

$$(z^2+4z+3)U(z) = \frac{z}{z-3} + z = \frac{z + z(z-3)}{(z-3)} = \frac{z^2-2z}{z-3}$$

$$U(z) = \frac{z^2-2z}{(z-3)(z^2+4z+3)} = \frac{z^2-2z}{(z-3)(z+1)(z+3)}$$

$$\frac{U(z)}{z} = \frac{z-2}{(z-3)(z+1)(z+3)} = \frac{A}{(z-3)} + \frac{B}{(z+1)} + \frac{C}{(z+3)}$$

$$z-2 = A(z+1)(z+3) + B(z-3)(z+3) + C(z-3)(z+1)$$

$z = 3 \Rightarrow 1 = A(4)(6) + 0 + 0$

$24A = 1 \Rightarrow A = \frac{1}{24}$

$z = -1 \Rightarrow -3 = 0 + B(-4)(2) + 0$

$$-3 = -8B \Rightarrow B = \frac{-3}{-8} = \frac{3}{8}$$

$$z = -3 \Rightarrow -5 = 0 + 0 + C(-6)(-2)$$

$$-5 = 12C \Rightarrow C = -\frac{5}{12}$$

$$\frac{U(z)}{z} = \frac{\frac{1}{24}}{z-3} + \frac{3/8}{z+1} + \frac{-5/12}{z+3}$$

$$U(z) = \frac{1}{24}\frac{z}{z-3} + \frac{3}{8}\frac{z}{z+1} - \frac{5}{12}\frac{z}{z+3}$$

$$U_n = Z^{-1}[U(z)] = \frac{1}{24}3^n + \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n$$

Q] $y_{n+2} - 5y_{n+1} + 6y_n = U_n$ $\quad y_0 = 0, y_1 = 1$
and U_n is the unit step function

Ans $U_n = \begin{cases} 1 & n = 0, 1, 2, 3 \dots \\ 0 & n < 0 \end{cases} = 1^n = 1$

$$\therefore Z[U_n] = Z[1] = \frac{z}{z-1}$$

$$Z[y_{n+2}] - 5Z[y_{n+1}] + 6Z[y_n] = Z[U_n]$$

$$z^2[Y(z) - y_0 - y_1 z^{-1}] - 5z[Y(z) - y_0] + 6Y(z) = \frac{z}{z-1}$$

$$z^2\left[Y(z) - 0 - \frac{1}{z}\right] - 5z[Y(z) - 0] + 6Y(z) = \frac{z}{z-1}$$

$$(z^2 - 5z + 6)Y(z) - z = \frac{z}{z-1}$$

$$(z^2 - 5z + 6)Y(z) = \frac{z}{z-1} + z = \frac{z + z(z-1)}{(z-1)} = \frac{z^2}{(z-1)}$$

$$Y(z) = \frac{z^2}{(z-1)(z^2 - 5z + 6)} = \frac{z^2}{(z-1)(z-2)(z-3)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} + \frac{C}{(z-3)}$$

$$z = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2)$$

$z = 1 \Rightarrow A = \frac{1}{2}$ $\qquad z = 2 \Rightarrow B = \frac{2}{-1} = -2$

$z = 3 \Rightarrow C = \frac{3}{2}$

$$\frac{Y(z)}{z} = \frac{1/2}{z-1} + \frac{-2}{z-2} + \frac{3/2}{z-3}$$

$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

$$Z^{-1}[Y(z)] = y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}3^n$$

Q] Solve $y_n + \frac{1}{4}y_{n-1} = U_n + \frac{1}{3}U_{n-1}$

where U_n is the unit step function.

$$Z[y_n] + \frac{1}{4}Z[y_{n-1}] = Z[U_n] + \frac{1}{3}Z[U_{n-1}]$$

$Z[U_{n-k}] = z^{-k}U(z)$ [Right shifting]

$$Y(z) + \frac{1}{4}z^{-1}Y(z) = \frac{z}{z-1} + z^{-1}\left(\frac{z}{z-1}\right)$$

[$\because$ $Z_n[U_n] = \frac{z}{z-1}$ as U_n is unit step function]

$$zY(z) + \frac{1}{4}Y(z) = \frac{1}{3}\frac{3z^2+z}{z-1}$$

$$\left(z+\frac{1}{4}\right)Y(z) = \frac{1}{3}\frac{3z^2+z}{z-1}$$

$$Y(z) = \frac{1}{3}\frac{3z^2+z}{(z-1)\left(z+\frac{1}{4}\right)}$$

$$\frac{Y(z)}{z} = \frac{1}{3}\left[\frac{3z+1}{(z-1)\left(z+\frac{1}{4}\right)}\right]$$

$$\frac{(3z+1)}{(z-1)\left(z+\frac{1}{4}\right)} = \frac{A}{(z-1)} + \frac{B}{\left(z+\frac{1}{4}\right)}$$

$$3z+1 = A\left(z+\frac{1}{4}\right) + B(z-1)$$

$$z=1 \Rightarrow 4 = A\left(\frac{5}{4}\right) \Rightarrow A = \frac{16}{5}$$

$$z = \frac{-1}{4} \Rightarrow \frac{1}{4} = B\left(\frac{-5}{4}\right) \Rightarrow B = \frac{-1}{5}$$

$$Y(z) = \frac{1}{3}\left[\frac{16}{5}\frac{z}{z-1} - \frac{1}{5}\frac{z}{z+\frac{1}{4}}\right]$$

$$y_n = \frac{1}{3}\left[\frac{16}{5} - \frac{1}{5}\left(\frac{-1}{4}\right)^n\right]$$