Course Title: STATISTICS AND DISCRETE MATHEMATICS
(Course Code: 23MA3BSSDM)

UNIT-2: PROBABILITY DISTRIBUTIONS

Poisson Distribution

The Poisson distribution can be used when the following conditions are met:

- Each trial may only have one of two outcomes: success or failure.
- Limiting process of the binomial distribution.
- The number of trials "n" tends to infinity.
- Probability of success "p" tends to zero.
- $np = \lambda$ is finite.
- Events are rare and independent, random and occur at a constant rate.
- Mean number of events is known

$$X \sim P(\lambda)$$

$$p(x) = \frac{e^{-\lambda x}\lambda^{x}}{x!}; x = 0,1,2,3,......$$

where x is the number of successes in the experiment

Parameters of the distribution: Mean $= \lambda$; Variance $= \lambda$

1. The traffic police recorded an average of 3 road accidents per week. The number of accidents is distributed according to a Poisson distribution. Calculate the probability in any week of exactly 2 accidents.

2. Alpha particles are emitted by radioactive source at the rate of three per every minute on the average. The number of particles is distributed according to the Poisson distribution. Calculate the probability of getting exactly 5 emissions in one minute.

3. A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 98% germination. Determine the probability that a particular packet will violate the guarantee.

4. It is known from past experience that in a certain plant there are on the average 4 industrial accidents per year. Find the probability that in a given year there will be less than 4 accidents.

5. In a town, 10 accidents take place in a span of 50 days. Assuming that the number of accidents follows the Poisson distribution, find the probability that there will be 3 or more accidents in a day.

6. A book contains 100 misprints distributed randomly throughout its 100 pages. Assuming Poisson distribution, find the probability that a page observed at random contains at leasttwo misprints.

7. A switch board can handle only 4 telephone calls per minute. If the incoming calls per minute follow a Poisson distribution with parameter 3, find the probability that the switchboard is over taxed in any one minute.

8. A car hire firm has two cars which it hires out day by day. The number of demands for acar on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days on which (i) there is no demand (ii) demand is refused.

9. A shop has 4 diesel generator sets which it hires every day. The demand for a generator set on an average is a Poisson variate with value 5/2. Obtain the probability that on a particularday (i) There was no demand (ii) A demand is refused.

10. A communication channel receives independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each micro second. Compute the probabilities of (i) no error during a micro second (ii) one error per micro second (iii)atleast one error per micro second (iv) two errors (v) At most two errors.

11. The probability that a news reader commits no mistake in reading the news is e^{-3} . Find the probability that on particular news broadcast he commits (i) only 2 mistakes (ii) more than 3 mistakes (iii) at most 3 mistakes.

12. In a certain factory turning out razors blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distributionto calculate the approximate number of packets containing (i) no defective (ii) one defective and (iii) two defective blades respectively in a consignment of 10,000 packets.

13. The number of accidents in a year to taxi in a city follows a Poisson distribution withmean 3. Out of 1000 taxi drivers find approximately the number of the drivers with (i) no accident in a year (ii) more than 3 accidents in a year.

14. 2% of the fuses manufactured by a firm are found to be defective. Find the probabilitythat a box containing 200 fuses contains (i) No defective fuses (ii) 3 or more defective fuses.

15. If the probability of a bad reaction from a certain injection is 0.001, determine the chances that out of 2000 individuals, (i) exactly 3 (ii) more than 2 will suffer a bad reaction.

16. A source of liquid is known to contain bacteria with the mean number of bacteria per cubic centimeter being equal to 3. Ten 1cc test tubes are filled with the liquid. Assuming that Poisson distribution is applicable calculate the probability that all the test tubes will show growth.

17. Fit a Poisson distribution for the following data and calculate the theoretical frequencies

x	0	1	2	3	4
f	111	63	22	3	1

18. Fit a Poisson distribution for the following frequency distribution.

x	0	1	2	3	4
f	46	38	22	9	1

19. Fit a Poisson distribution for the following frequency distribution.

x	0	1	2	3	4
f	122	60	15	2	1

20. The frequency of accidents per shift in a factory is as shown in the following table:

Accidents per shifts	0	1	2	3	4
Frequency	180	92	24	3	1

Calculate the mean number of accidents per shifts and the corresponding Poisson distribution and compare with actual observations.

21. The number of accidents per day (x) as recorded in a textile industry over a period of 400days is given. Fit a Poisson distribution for the data and calculate the theoretical frequencies.

x	0	1	2	3	4	5
f	173	168	37	18	3	1

Geometric Distribution

Geometric distributions are based on three key assumptions:

- Each trial may only have one of two outcomes: success or failure.
- For each trial, the success probability, represented by p, is the same.

$$X \sim G(p)$$

$$p(x) = pq^{x-1}; x = 1, 2, 3, \ldots\ldots$$

where x is the trial on which the first success occurs

Cumulative Distribution Function: $P(X \leq x) = 1 - q^x$

Parameters of the distribution: Mean $= \frac{1}{p}$; Variance $= \frac{1-p}{p^2}$

1. The probability that a phone call leads to a sale is 0.4. Calculate the probability that the first sale occurs in the fifth call.

2. In a game of darts, the probability of missing the target is 0.6. What is the probability that the first bullseye is hit in the third try?

3. A representative from the National Football League's Marketing Division randomly selects people on a random street in Kansas City, Kansas until he finds a person who attended the last home football game. Let the probability that he succeeds in finding such a person is 0.20. And, let X denote the number of people he selects until he finds his first success.
   (i) What is the probability that the marketing representative must select 4 people before he finds one who attended the last home football game?
   (ii) What is the probability that the marketing representative must select more than 6 people before he finds one who attended the last home football game?
   (iii) How many people should we expect (that is, what is the average number) the marketing representative needs to select before he finds one who attended the last home football game? And, while we're at it, what is the variance?

4. An old lawn mower has a 20% chance of starting on a particular pull. Find the probability that it takes (i) exactly 3 pulls to start the mower (ii) 10 or fewer pulls to start the mower

5. A person decides to continue placing a bet of Rs.5000 on the number 5 in consecutive spins of a roulette wheel until he wins. On any spin there is a 1 on 50 chances that the ball would land on the number 5.
   (i) How many spins do you expect until he wins?
   (ii) What is the amount he is expected to spend until he has his first win?

(iii) What are the chances that it takes 5 spins before he wins?
(iv) What are the chances that it would take him more than 50 chances to win?

6. To win a board game, a person A needs a sum of 4 with two dice. What is the probability that it takes A under 5 tries to win? How many rolls would you expect A to take until she wins?

7. A practicing shooter scores 93% of his shots during a training session. What are the chances that he would not miss a single shoot till his 20^{th} try? What is the expected number of shots taken before his first miss?

Uniform Distribution (Continuous)

Continuous uniform distributions, also known as rectangular distributions, are probability distributions where the probability density function (PDF) is constant within a certain interval and zero elsewhere. This means that all outcomes within the interval are equally likely.

$$X \sim U(a,b)$$

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ o & otherwise \end{cases}$$

Cumulative Distribution Function: $P(X \le x) = \frac{x-a}{b-a}, x \in [a,b]$

Parameters of the distribution: Mean $= \frac{b-a}{2}$; Variance $= \frac{(b-a)^2}{12}$

1. If X is uniformly distributed in $-2 \le X \le 2$, find (i) $P(X < 1)$ (ii) $P\left(|X-1| \ge \frac{1}{2}\right)$.

2. If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$, find $P(X < 0)$.

3. A random variable X has a uniform distribution $U[-3,3]$. Find k if $P(X > 3) = k$.

4. A point is chosen at random from the line segment[0,2]. What is the probability that the chosen point lies (a) $1 \le X \le \frac{3}{2}$ (b) $X \le 1$ (c) $X \ge 3$

5. If X is uniformly distributed in $[-\alpha, \alpha]$ with $\alpha > 0$ such that $P(X > 1) = \frac{1}{3}$ then determine α.

6. A bus travels between two cities A and B which are 100 miles apart. If the bus has a breakdown, the distance X of the point of breakdown from the city A has a uniform distribution $U[0,100]$. There are service garages in the city A, city B and midway between the two cities such that in case of a breakdown a tow truck is sent from the garage nearest to the point of breakdown.
(i) What is the probability that the tow truck has to travel more than 10 miles to reach the bus?
(ii) Would it be more "efficient" if the three service garages were placed at 25, 50 and 75 miles from city A, apart from service garages at city A and city B?

7. If a conference room cannot be reserved for more than 4 hours, find the probability that a given conference lasts more than 3 hours.

8. The daily amount of coffee (in liters) dispensed by a machine is uniformly distributed with $a = 7$ and $b = 10$. Determine the probability that the amount of coffee dispensed by the machine will be (i) at most 8.8 liters (ii) more than 7.4 liters but less than 9.5 liters (iii) a least 8.5 liters.

9. The driving time X from a person's home to the train station is uniformly distributed as $U[10,50]$. If it takes 2 minutes to board the train, determine the probability that the person caches the 7 am train if he starts at 6:43am from his home.

10. A bus arrives every 10 minutes at a bus stop. Assuming that the waiting time X for a bus is uniformly distributed, find the probability that the person has to wait for the bus (i) for more than 7 minutes (ii) between 2 and 7 minutes

11. The amount charged for a visit to a dental clinic is uniformly distributed from 0 to 1000 (in INR). Given that the amount charged for a visit exceeds Rs 500, calculate the probability that it exceeds Rs. 750.

Exponential Distribution

The exponential distribution is a probability distribution used to model the time between events or arrivals, or the lifetime of a system

$$X \sim E(\alpha)$$

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0, \alpha > 0 \\ 0 & otherwise \end{cases}$$

Cumulative Distribution Function: $P(X \leq x) = 1 - e^{-\alpha x}$

Parameters of the distribution: Mean $= \frac{1}{\alpha}$; Variance $= \frac{1}{\alpha^2}$

1. In a certain town, the duration of shower is exponentially distributed with mean 5 minutes. What is the probability that the shower will last for (i) less than 10 minutes (ii) 10 minutes or more and (iii) between 10 and 12 minutes.

2. The length of a telephone conversation in a booth is exponentially distributed and found on an average to be 5 minutes. Find the probability that the random call made from this booth (i) Ends in less than 5 minutes (ii) Between 5 and 10 minutes.

3. At a certain city bus stop, three buses arrive per hour on an average. Assuming that the time between successive arrivals is exponentially distributed; find the probability thatthe time between the arrivals of successive buses is (i) less than 10 minutes (ii) at least 30 minutes.

4. The sales per day in a shop is exponentially distributed with average sale amounting to Rs100/- and net profit is 8%. Find the probability that the net profit exceeds Rs. 30/- on2 consecutive days.

5. The daily turnover in a medical shop is exponentially distributed with Rs.6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs.500on a randomly chosen day.

6. Increase in sales per day in a shop is exponentially distributed with mean of Rs.600/-. Sales tax is to be levied at 9%. What is the probability that sales tax will exceed Rs.81per day?

7. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is (i) Less than 200 months (ii) Between 100 months and 25years?

8. The time X (seconds) that it takes a certain online computer terminal (the elapsed time between the end of user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected time 20 seconds. Compute the probabilities (a) $P(X \leq 30)$ (b) $P(X \geq 20)$ (c) $P(20 \leq X \leq 30)$.

9. Let the mileage (in thousands of miles) of a particular tyre be a random variable X having the probability density $f(x) = \begin{cases} \frac{1}{20} e^{-x/20} & x > 0 \\ 0 & x \leq 0 \end{cases}$.Find the probability that one of these tyres will last (i) at most 10,000 miles (ii) anywhere from16,000 to 24,000 miles (iii) at least 30,000 miles (iv) mean (v) variance

Normal Distribution/ Gaussian Distribution

$$X \sim N(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} ; -\infty < x, \mu < \infty$$

Parameters of the distribution: Mean $= \alpha$; Variance $= \sigma^2$

1. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and S.D of 60 hours. estimate thenumber of bulbs likely to burn for (a) More than 2150 hours (b) Less than 1950 hours(c) More than 1920 hours and but less than 2160 hours.

2. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find themean and S.D. of the distribution.

3. If the total cholesterol values for a certain population are approximately normally distributed with a mean of 200mg/ml and standard deviation of 20mg/ml. Find the probability that an individual selected at random from this population will have a cholesterol value:
   i. Between 180 and 200mg/ml.
   ii. Greater than 225mg/ml.
   iii. Less than 150mg/ml.

4. In a certain examination, the percentage of candidates passing and getting distinctions were 45 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively.

5. A manufacturer of air-mail envelopes knows from experience that weight of the envelopes is normally distributed with mean 1.95 gm and S.D. 0.05 gm. About how many envelopes weighting (i) 2 gm or more (ii) 2.05 gm or more can be expected in agiven packet of 100 envelopes.

6. The mean height of 500 students is 151 cm and the S.D. is 15 cm. Assuming that the heights are normally distributed, find how many student's heights lie between 120 and 155 cm.

7. The mean and S.D. of the marks obtained by 1000 students in an examination are respectively 34.4 and 16.5. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 and 60.

8. In a examination taken by 500 candidates, the average and S.D. of marks obtained (normally distributed) are 40% and 10%. Find approximately (i) how many will pass, if 50% is fixed as a minimum? (ii)What should be the minimum if 350 candidates are to pass? (iii) How many have scored marks above 60%?

9. The mean inside diameter of a sample of 200 washers produced by a machine is 5.02 mmand the S.D. is 0.05 mm. the purpose for which these washers are intended allows a maximum tolerance in the diameter of 4.96 to 5.08 mm, otherwise the washers are considered defective. Determine the percentage of defective washers produced by the machine, assuming the diameters are normally distributed.

10. It is given that the age of thermostats of particular makes follow the normal law with mean 5 years and S.D. 2 years. 1000 units are sold out every month. How many of them will have to be replaced at the end of the second year?

11. The income of a group of 10,000 persons was found to be normally distributed with meanRs. 750 p.m. and S.D. of Rs. 50. Show that, of this group, about 95%had income exceeding Rs. 668 and only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100.

12. If the top 15% of the students receives A grade and bottom 10% receives F grades in a mathematics examination, determine the (a) minimum marks to get an A grade (b) minimum mark to pass. Assume that the marks are normally distributed with mean 76 and standard deviation 15.

Gamma Distribution

$$X \sim \Gamma(\alpha, \beta)$$

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)} & x > 0 \\ 0 & otherwise \end{cases}$$

Cumulative Distribution Function: $P(X \leq x) = \frac{\Gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)}$

Parameters of the distribution: Mean $= \alpha\beta$; Variance $= \alpha\beta^2$

1. The daily consumption of electric power (in million Kw-hours) in a certain city is a random variable X having the probability density

$$f(x) = \begin{cases} \frac{1}{9} x e^{-x/3}, if\ x > 0 \\ 0, if\ x \leq 0 \end{cases}$$

Find the probability that the power supply is adequate on any given day if the capacity of the power plant is 12 million Kw hours.

2. The lifetime X (in months) of a computer has a gamma distribution with mean 24 months and standard deviation 12 months. Find the probability that the computer will (i) last between 12 and 24 months (ii) last at most 24 months.

3. Suppose that the time (in hours) taken by a homeowner to mow his lawns is a random variable

having a gamma distribution with parameters $\alpha = 2$ and $\beta = 2$. Find the probability that it takes (i) at most 1 hour (ii) at least 2 hours (iii) between 0.5 to 1.5 hours to mow the lawn.

4. The survival time (in weeks) of a male mouse exposed to radiation has a gamma distribution with $\alpha = 8$ and $\beta = 15$. Find the probability that the mouse survives (i) between 60 and 120 weeks (ii) at least 30 weeks. Find the mean and variance of X.

5. If a random variable has gamma distribution with $\alpha = 2$ and $\beta = 2$. Find (i) mean (ii) standard deviation (iii) the probability that X will take a value less than 4.

6. An actuary models the occurrence of claims from a portfolio of insurance policies. The time until the occurrence of the second claim is modelled by a gamma distribution with mean 10 minutes and variance 50 minutes. Determine the parameters of the distribution. Hence determine the probability that the time until the occurrence of the second claim exceeds 20 minutes.

7. Daily consumption of milk in a town in excess of 20,000 litres is approximately given by Gamma distribution with $\alpha = 3$ and $\beta = 10,000$. The town has a daily stock of 30,000 litres. Find the probability that the stock is insufficient on a given day.

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UNIT-2
PROBABILITY DISTRIBUTION

★ Random variable: A random variable is a function that assigns a real number to every sample point in the sample space of a random experiment.
Random variables are usually denoted by X, Y, Z...

NOTE: Different random variables may be associated with same sample space S.

- The set of all real numbers of random variables X is called range of X.

Eg:- Consider the random experiment of tossing a coin.

$$S = \{H, T\}$$

$$f: S \to R \text{ defined by } f(s) = \begin{cases} 1 & \text{if } s = H \\ 0 & \text{if } s = T \end{cases}$$

$f(s) = X$ is a random variable on S & range of X is $\{1, 0\}$.

★ Discrete Random Variable: If a random variable takes finite or countably infinite no. of values, it is called discrete random variable.

Eg:- • Tossing a coin
• No. of students in on class
• No. of people entering a queue

★ Continuous Random variable: If a random variable takes non-countable infinite no. of values then it is non-discrete or continuous random variable. If range of random variable X is an interval of real numbers, then X is continuous random variable.

Eg:- • Weight of students in a class
• Pointer of a speedometer

★ Discrete Probability Distribution [Probability Mass Function]:

If for each x_i of discrete random variable x, we assign a real number $p(x_i)$ such that $p(x_i) \geq 0$ & $\sum p(x_i) = 1$ then $p(x_i)$ is called probability function.

The set of values $[x_i, p(x_i)]$ is called a discrete probability distribution of discrete random variable X.

★ Bernoulli Trial (Has only two outcomes, success & failure.)

The random experiment which results in either success or failure is called Bernoulli trial. Repeating a Bernoulli trial n times results in Binomial Distribution.

Eg:- Tossing a coin.

★ Poisson Distribution: This is regarded as limiting form of binomial distribution. When $n \to \infty$ & $p \to 0$, $np \to \lambda$

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

NOTE: • $p(x) \geq 0$ & $\sum_{x=0}^{\infty} p(x) = e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots\right) = e^{-\lambda} e^{\lambda} = 1$

• Therefore Poisson Distribution is a probability distribution.

★ Problems:-

1. In a certain factory manufacturing electric bulbs, there is a chance of 1/500 for any bulbs to be defective. The bulbs are packede in packets of 50. Use the Poisson distribution to calculate the approximate number of packets containing no defective, 1, 2, 3 defective bulb in a consignment of 10,000 packets.

ans $x \to$ no. of bulbs which are defective per packet

$p = \frac{1}{500}$ $\quad n = 50$ $\quad \lambda \to$ avg. no. of defective bulbs/packet

$\lambda = np = \frac{1}{500} \times 50 = \frac{1}{10} = 0.1$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.1}(0.1)^x}{x!}$$

1) $P(x=0) = \frac{e^{-0.1}(0.1)^0}{0!} = e^{-0.1} = 0.9048$

No. of packets with zero defective bulbs $= 0.9048 \times 10000$
$\simeq \underline{9048}$

2) $P(x=1) = \frac{e^{-0.1}(0.1)^1}{1!} = 0.0904$

No. of packets with 1 defective bulb $= 0.09048 \times 10000 \simeq 904.8$
$\simeq \underline{905}$

3) $P(x=2) = \frac{e^{-0.1}(0.1)^2}{2!} = \frac{0.9048 \times 0.01}{2} = 0.0045$

No. of packets with 2 defective bulbs $= 0.0045 \times 10000$
$\simeq \underline{45}$

4) $P(x=3) = \frac{e^{-0.1}(0.1)^3}{3!} = \frac{0.9048 \times 0.001}{3 \times 2} = 1.508 \times 10^{-4}$

No. of packets with 3 defective bulbs $= 1.508 \times 10^{-4} \times 10000$
$= 1.508 \simeq \underline{2}$

2. In a book of 800 pages, 300 typographical errors are noticed. Assuming Poisson law for the number of errors per page, find the probability that a randomly chosen 4 pages will contain no errors.

ans $x \to$ no. of errors for 4 pages

$$\lambda = \frac{300}{800} \times 4 = 3/8 \times 4 = 3/2$$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3/2}(3/2)^x}{x!}$$

$P(x=0) = e^{-3/2} = \cancel{0.6872}$ $\underline{0.2231}$

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3. The no. of emergency admissions each day to a hospital is found to have Poisson distribution with mean 4. Find the probability that on a particular day there will be no emergency admissions.

ans $x \to$ no. of emergency admissions per day

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad mean = 4 = \lambda$$

$$P(x=0) = \frac{e^{-4}4^{0}}{0!} = \underline{0.0183}$$

4. A car hire firm has two cars which it hires daily. The no. of demands for a car on each day is distributed as a Poisson variate with a mean of 1.5. Obtain the proportion of days on which
i) there was no demand ii) Demands were refused.

ans $x \rightarrow$ no. of demands (for a car each day)

$$\lambda = 1.5 \; ; \; P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-1.5}(1.5)^{x}}{x!}$$

i) $P(x=0) = e^{-1.5} = \underline{0.2231}$

ii) $P(x>2) = 1 - P(x \leq 2) = 1 - [P(0) + P(1) + P(2)]$

$$= 1 - e^{-1.5}\left[1 + (1.5) + \frac{(1.5)^2}{2}\right]$$

$$= \underline{0.1912}$$

5. A distributor of bean seeds determines from extensive tests that 5% of a large batch of seeds will not germinate. He sells the seeds in packets of 200 & guarantees 98% germination. Determine the probability that a particular packet will violate the guarantee.

ans $x \rightarrow$ no. of seeds that will not germinate per packet

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \; ; \; \lambda = np \; ; \; n = 200 \; ; \; p = 5\%$$

$$\lambda = 200 \times \frac{5}{100} = 10$$

$$P(x) = \frac{e^{-10}(10)^{x}}{x!}$$

According to his claim, 98% of 200, i.e. 196 seeds will germinate & 4 will not.

$$P(x>4) = 1 - P(x \leq 4)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4)]$$

$$= 1 - e^{-10}\left[1 + 10 + \frac{(10)^2}{2} + \frac{(10)^3}{6} + \frac{(10)^4}{24}\right]$$

$$= \underline{\underline{0.9707}}$$

6. Fit a Poisson distribution to the following data & calculate the theoretical frequencies

x	0	1	2	3	4
f	109	65	22	3	1

ans $\lambda = \dfrac{\Sigma f_i x_i}{\Sigma f_i (200)} = \dfrac{65+44+9+4}{109+65+22+3+1} = \dfrac{122}{200} = 0.61$

$$P(x) = \frac{e^{-0.61}(0.61)^x}{x!}$$

x	f_i	$P(x)$	Expected frequency	
0	109	0.5433	$0.5433 \times 200 \simeq 108.66$	$\simeq 109$
1	65	0.3314	$0.3314 \times 200 \simeq 66.28$	$\simeq 66$
2	22	0.1010	$0.1010 \times 200 \simeq 20.2$	$\simeq 20$
3	3	0.0205	$0.0205 \times 200 \simeq 4.1$	$\simeq 4$
4	1	0.00313	$0.00313 \times 200 \simeq 0.626$	$\simeq 1$
				200

make adjustments to make this possible

sum shd always be Σf_i

7. The no. of accidents per day (x) as recorded in a textile industry over a period of 400 days is given. Fit a Poisson distribution for the data & calculate the theoretical frequencies.

x	0	1	2	3	4	5
f	173	168	37	18	3	1

ans $\lambda = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{168+74+54+12+5}{173+168+37+18+3+1} = \dfrac{313}{400} = 0.7825$

$$P(x) \simeq e^{-0.7825} \frac{(0.7825)^x}{x!}$$

x	f	$P(x)$	Expected frequency	
0	173	0.4573	$400 \times 0.4573 \simeq 182.92$	~ 183
1	168	0.3578	$400 \times 0.3578 \simeq 143.12$	~ 143
2	37	0.1399	$400 \times 0.1399 \simeq 55.96$	~ 56
3	18	0.0365	$400 \times 0.0365 \simeq 14.6$	~ 15
4	3	0.00714	$400 \times 0.00714 \simeq 2.856$	~ 3
5	1	0.00118	$400 \times 0.00118 \simeq 0.472$	~ 0
				400

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☆ **Geometric Distribution:** It is a probability distribution that models the no. of trials required to achieve the first success in a sequence of independent Bernoulli trials, where each trial has a constant probability of success.

The probability function for this distribution will be denoted by $P(X) = p(1-p)^{x-1}$. This probability function is called the geometric probability function & the corresponding distribution is called the geometric probability distribution.

Mean & Variance of Geometric distribution:

Mean, $\mu = \frac{1}{p}$

Variance, $\sigma^2 = \frac{q}{p^2}$

☆ Problems:

1. If the probability that a target is destroyed on any one shot is 1/3. What is the probability that it would be destroyed in the third shot not before?

ans $p = 1/3$ [probability of destruction in one shot]

$P(x) = p(1-p)^{x-1}$

$P(x=3) = qqp = p(1-p)^2 = \frac{1}{3}\left(\frac{2}{3}\right)^2 = \frac{4}{27} = 0.1481$

x starts from 1

2. The probability that the prediction of a soothsayer will come true is 0.01. What is the probability that his 13th prediction is the first one to be true?

ans $p = 0.01$ [Probability of prediction to be true]

$P(x) = p(1-p)^{x-1} = p(1-p)^{12}$

$P(x=13) = p(1-p)^{12} = (0.01)(0.99)^{12} = \underline{8.86 \times 10^{-3}}$

3. The probability that a thief is caught while stealing is 0.65. What is the probability that he is caught at the (i) 5th attempt & not before (ii) 5th attempt or earlier (iii) 5th attempt or later.

ans $p = 0.65$ (Probability of getting caught)

$P(x) = p(1-p)^{x-1}$

(i) $P(x=5) = p(1-p)^4 = (0.65)(0.35)^4 = \underline{9.75 \times 10^{-3}}$

(ii) $P(1) + P(2) + P(3) + P(4) + P(5)$

$= p + pq + pq^2 + pq^3 + pq^4$

$= 0.65(1 + 0.35 + (0.35)^2 + (0.35)^3 + (0.35)^4)$

$= \underline{0.9947}$

$qqqqp = q^4p \Rightarrow P(5)$
$qqqp = q^3p \Rightarrow P(4)$
$qqp = q^2p \Rightarrow P(3)$
$qp \Rightarrow P(2)$
$p \Rightarrow P(1)$

(iii) $1 - P(x<5) = 1 - [P(x=1) + P(x=2) + P(x=3) + P(x=4)]$

$1 - [p + pq + pq^2 + pq^3]$

$= 1 - (0.65)[1 + 0.35 + (0.35)^2 + (0.35)^3]$

$= \underline{0.015}$

$qqqqp \ldots$ $P(x \geq 5)$

4. In a certain city, the probability that rain occurs on a day during June is 5/8. Find the probability that there is rain on June 5th & not earlier.

ans $p = 5/8$ (Probability of rain in June)

$P(x) = p(1-p)^{x-1}$

p
qp
qqp
$qqqp$
$qqqqp = pq^4$

$P(x=5) = p(1-p)^4 = 5/8\,(3/8)^4 = \underline{0.0123}$

5. For a boy pelting stones at a target, the probability of a successful hit is 0.4. What is the probability that at least 5 trials are required for him to have the first successful hit?

ans $p = 0.4$ (Probability of successful hit)

$P(x) = p\,(1-p)^{x-1}$

$P(x \geq 5) = 1 - P(x<5) = 1 - [P(x=1)+P(x=2)+P(x=3)+P(x=4)]$

$= 1 - (p + pq + pq^2 + pq^3)$ ~~pq^4~~

$= 1 - 0.4[1 + 0.6 + (0.6)^2 + (0.6)^3]$

$= \underline{0.1296}$

6. An expert shooter can hit a target 95% of the time. Find the probability that he will hit the continuously 14 times & will miss it at the 15th attempt.

ans $p = 1 - 0.95$ (Probability of not hitting target), $q = 0.95$ (Probability of hitting target)

$p = 0.05$ $\qquad pq^{14}$

$P(x) = p(1-p)^{x-1}$

$P(x=15) = p(1-p)^{14} = 0.05(1-0.05)^{14} = \underline{0.0243}$

7. The lifetime risk of developing pancreatic cancer is about one in 78 (1.28%). Let X = the no. of people you ask before one says he or she has pancreatic cancer. The random variable X in this case includes only the no. of trials that were failures & does not count the trial that was a success in finding a person who had the disease. ~~The appropriate formula for this~~ ~~so~~ X is a discrete random variable with a geometric distribution.

(i) What is the probability that you ask 9 people before one says he or she has pancreatic cancer? This is asking, what is the probability that you ask 9 people unsuccessfully and the tenth person is a success ?

(ii) What is the probability that you must ask 20 people ?

ans. $p = 1/78$ (Probability of having pancreatic cancer)

(i) $P(x=10) = q\,q\,q \cdots q\,p$

$= q^9 p$

$P(x) = p(1-p)^{x-1}$

$P(x=10) = pq^9 = p(1-p)^{10-1} = \frac{1}{78}\left(1-\frac{1}{78}\right)^9 = \underline{\underline{0.0114}}$

(ii) $P(x=20) = p(1-p)^{20-1} = \frac{1}{78}\left(1-\frac{1}{78}\right)^{19} = \underline{\underline{0.01003}}$

☆ Continuous Probability distribution:-

→ If for every x belonging to range of a continuous random variable X, we assign a real number $f(x)$ satisfying the condition $f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Then $f(x)$ is called continuous probability function or Probability density function.

NOTE: i) If (a, b) is a subinterval of range space of X then the probability that x lies in (a, b) is defined to be integral of $f(x)$ between a & b

$$p(a < x < b) = \int_{a}^{b} f(x)\,dx$$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ geometrically means that the area bounded by the curve $f(x)$ & x-axis is 1

☆ Mean & Variance for continuous probability distribution:

If X is a continuous random variable with PDF $f(x)$

where, $-\infty < x < \infty$, the mean μ & variance σ^2 of X is defined as:

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} f(x)(x-\mu)^2\, dx$$

1. Exponential Distribution:-

If $f(x) = \begin{cases} \alpha e^{-\alpha x} & x>0 \\ 0 & \text{otherwise} \end{cases}$, where $\alpha > 0$, then it is called exponential distribution.

→ Clearly $f(x)>0$ & $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \alpha e^{-\alpha x}\, dx = 1$

→ Thus $f(x)$ satisfies both the conditions for continuous probability function.

★ Mean & Variance of Exponential Distribution :-

Mean, $\mu = \frac{1}{\alpha}$

Variance, $\sigma^2 = \frac{1}{\alpha}$

SD, $\sigma_s = \frac{1}{\sqrt{\alpha}}$

★ Problems:-

1. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is (i) Less than 200 months (ii) Between 100 months & 25 years?

ans $x \Rightarrow$ life of compressor in months

$f(x) = \begin{cases} \alpha e^{-\alpha x} & x>0 \\ 0 & \text{otherwise} \end{cases}$

mean $= 200 = \frac{1}{\alpha}$; $\alpha = \frac{1}{200}$

i) $P(x<200) = \int_0^{200} \alpha e^{-\alpha x}\, dx = \int_0^{200} \frac{1}{200} e^{-x/200}\, dx$

$$= \frac{1}{200}\left[\frac{e^{-x/200}}{-1/200}\right]_0^{200} = -[e^{-1} - e^{-0}]$$

$$= -[e^{-1} - 1]$$

$$= 1 - e^{-1}$$

$$= 0.6321$$

ii) $P(100 < x < 300) = \int_{100}^{300} \frac{1}{200} e^{-x/200} dx = \frac{1}{200}\left[\frac{e^{-x/200}}{-1/200}\right]$

$= \left[-e^{-x/200}\right]_{100}^{300} = -\left[e^{-1.5} - e^{-0.5}\right] = \underline{0.3834}$

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2. If the life time of a certain type of electric bulbs is distributed as an exponential variate with mean of 1000 hrs, what is the probability that a bulb will last for more than 1500 hours? If two bulbs are selected at random find the probability that (i) both the bulbs (ii) at least one bulb will last for more than 1500 hours.

ans $x \to$ life time of a bulb in hrs

mean = 1000 hrs = $\frac{1}{\alpha}$; $\alpha = \frac{1}{100}$; $f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{Otherwise} \end{cases}$

$\cancel{f(x)} =$

$P(x>1500) = \int_{1500}^{\infty} \alpha e^{-\alpha x} dx = \frac{1}{1000}\int_{1500}^{\infty} e^{-x/1000} dx$

$= \frac{1}{1000}\left[\frac{e^{-x/1000}}{-1/1000}\right]_{1500}^{\infty} = -\left[e^{-\infty} - e^{-3/2}\right] = e^{-3/2}$

$= \underline{0.2231}$

i) $P(A \cap B) = P(A)P(B) = 0.2231 \times 0.2231 = \underline{0.0497}$

ii) $P(A) + P(B) - P(A \cap B) = P + P - P^2 = 2(0.2231) - 0.0497$

$= \underline{0.3965}$

3. Students arrive at a restaurant according to an approximate exponential process at a mean rate of 30 students per hour. What is the probability that the bouncer has to wait more than 3 minutes to card the next student?

ans $f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

~~α = 30 students/hr~~ $x \to$ time interval between arrival of two students in min mean = 2 min (1 student comes every 2 mins) $\left\{\frac{30}{60}=\frac{1}{2}\right\}$

$\alpha=\frac{1}{2}$

$$P(x>3)=\int_3^\infty \frac{1}{2}e^{-x/2}\,dx=\frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_3^\infty=-\left[e^{-\infty}-e^{-3/2}\right]=e^{-3/2}$$

$$=\underline{0.2231}$$

4. The increase in sales per day in a shop is exponentially distributed with Rs. 800 as the average. If sales tax is levied at the rate of 6%, find the probability that the increase in sales tax return from that shop will exceed ₹30 per day.

ans $x \rightarrow$ Increase in sales tax return per day

mean $=800\times\frac{6}{100}=48$; mean $=\frac{1}{\alpha} \Rightarrow \alpha=\frac{1}{48}$

$$f(x)=\begin{cases}\alpha e^{-\alpha x} & x>0\\ 0 & \text{otherwise}\end{cases}$$

(The increase in sales per day is ₹800 & sales tax leveid is at rate of 6%. ∴ mean = 48)

$$P(x>30)=\int_{30}^\infty \alpha e^{-\alpha x}\,dx=\frac{1}{48}\int_{30}^\infty e^{-x/48}\,dx=-\left[e^{-\infty}-e^{-\frac{30}{48}}\right]$$

$$=\underline{0.5352}$$

5. The number of days ahead travellers purchase their airline tickets can be modelled by an exponential distribution with the average amount of time equal to 15 days. Find the probability that a traveller will purchase a ticket fewer than ten days in advance.

ans $x \rightarrow$ No. of days in advance a traveller purchases ticket

mean = 15

$\alpha=\frac{1}{15}$; $f(x)=\begin{cases}\alpha e^{-\alpha x} & x>0\\ 0 & \text{otherwise}\end{cases}$

$$P(x<10)=\int_0^{10}\alpha e^{-\alpha x}\,dx=\frac{1}{15}\int_0^{10}e^{-x/15}\,dx=-\left[e^{-10/15}-e^{-0}\right]$$

$$=-e^{-10/15}=\underline{0.4865}$$

2. Normal Distribution :- Both mean & SD will be given in Ques

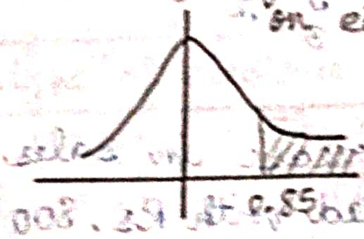
If $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where $-\infty < x < \infty$, $-\infty < \mu < \infty$ & $\sigma > 0$ is called Normal distribution

Total area of curve is 1, on either side 0.5

Eg:- 1) $P(z > 0.85) = 0.5 - \phi(0.85)$

$= 0.5 - 0.3023$

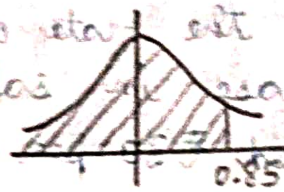
$= 0.1977$



2) $P(z < 0.85) = 0.5 + \phi(0.85)$

$= 0.5 + 0.3023$

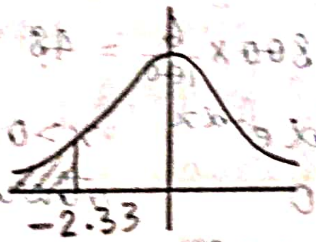
$= 0.8023$



3) $P(z < -2.33) = 0.5 - \phi(2.33)$

$= 0.5 - 0.4901$

$= 0.0099$

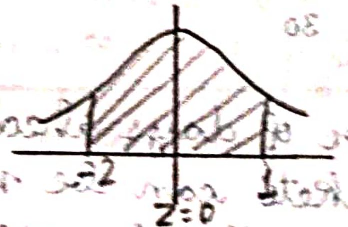


4) $P(-2 < z < 1)$

$= \phi(2) + \phi(1)$

$= 0.4772 + 0.3413$

$= 0.8185$



★ Problems :-

1. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours & S.D of 60 hours. Estimate the no. of bulbs likely to burn for

a) More than 2150 hours b) Less than 1950 hours

c) More than 1920 hours, but less than 2160 hours

ans $x \to$ life of a bulb in hours

$\mu = 2040$ & $SD(\sigma) = 60$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$$

P for one bulb ← $\underbrace{0.0336+0.0336+\ldots+0.0336}_{2000 \text{ times}} = 2000\times0.0336$

a) $P(x>2150) = P(z>1.83)$

$= 0.5-\phi(1.83)$

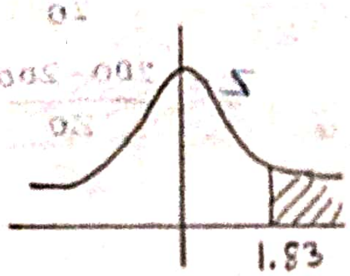
$= 0.5-0.4664$

$= \underline{0.0336}$

$z = \frac{x-2040}{60}$

$z = \frac{2150-2040}{60}$

$z = 1.83$



No. of bulbs $= 2000\times0.0336 = 67.2 \simeq \underline{67}$.

b) $P(x<1950) = P(z<-1.5)$

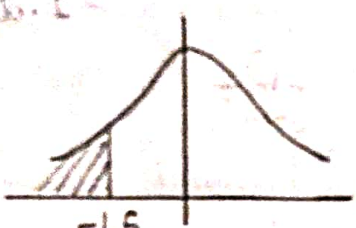
$= 0.5-\phi(1.5)$

$= 0.5-0.4332$

$= 0.0668$

$z = \frac{1950-2040}{60}$

$z = -1.5$



No. of bulbs $= 2000\times0.0668 = 133.6 \simeq \underline{\underline{134}}$.

c) $P(1920<x<2160) = P(-2<z<2)$

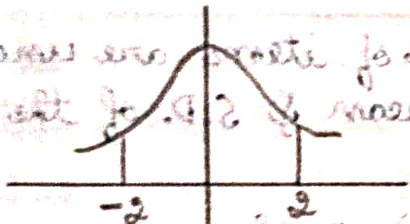
$= 2\phi(2)$

$= 2(0.4772)$

$= 0.9544$

$z = \frac{1920-2040}{60} = -2$

$z = \frac{2160-2040}{60} = 2$



No. of bulbs $= 2000\times0.9544 = 1908.8 \simeq \underline{1909}$

2. If the total cholesterol values for a certain population are approximately normally distributed with a mean of 200 mg/ml & standard deviation of 20 mg/ml. Find the probability of that an individual selected at random from this population will have a cholesterol value:

i) Between 180 & 200 mg/ml.

ii) Greater than 225 mg/ml

iii) Less than 150 mg/ml

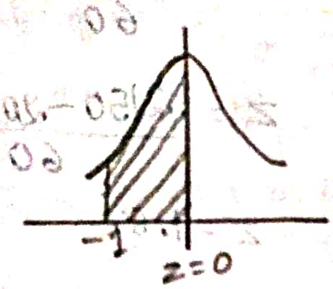
ans $x \to$ Cholesterol of a person

$\mu = 200, \sigma = 20$

$z = \frac{x-\mu}{\sigma} = \frac{x-200}{20}$

i) $P(180 < x < 200) = P(-1 < z < 0)$ $\quad Z = \frac{180-200}{20} = -1$

$= \phi(1) = \underline{\underline{0.3413}}$ $\quad Z = \frac{200-200}{20} = 0$

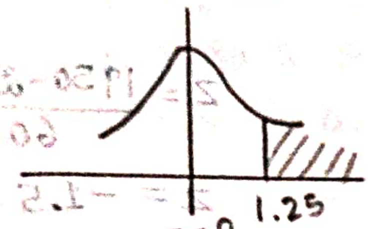


ii) $P(x > 225) = P(z > 1.25)$ $\quad Z = \frac{225-200}{20} = 1.25$

$= 0.5 - \phi(1.25)$

$= 0.5 - 0.3944$

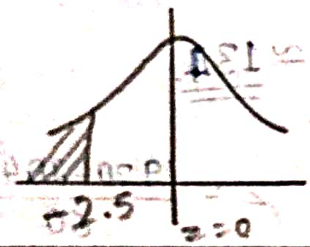
$= \underline{\underline{0.1056}}$



iii) $P(x < 150) = P(z < -2.5) = 0.5 - \phi(2.5)$ $\quad Z = \frac{150-200}{20} = -2.5$

$= 0.5 - 0.4938$

$= \underline{\underline{0.0062}}$



*3. In a normal distribution, 31% of items are under 45 & 8% are over 64. Find the mean & S.D. of the distribution.

ans Given: $P(x<45) = 0.31$ & $P(x>64) = 0.08$

$\mu = ?$, $\sigma = ?$

$Z = \frac{x-\mu}{\sigma}$

$P(x<45) = 0.31$ $\quad \left\{ \begin{array}{c} x<45 \\ x-\mu < 45-\mu \\ \overline{\sigma} \quad\quad \overline{\sigma} \\ z < Z_1 \end{array} \right\}$

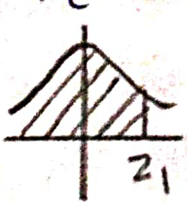
$P(z<z_1) = 0.31$, $z_1 = \frac{45-\mu}{\sigma}$

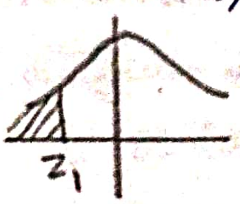
$P(z>z_2) = 0.08$, $z_2 = \frac{64-\mu}{\sigma}$

$P(z<z_1) = 0.31$ $\quad \Rightarrow 0.5 - \phi(z_1) = 0.31$

$\phi(z_1) = 0.19$

$\therefore z_1 = \underline{\underline{-0.5}}$

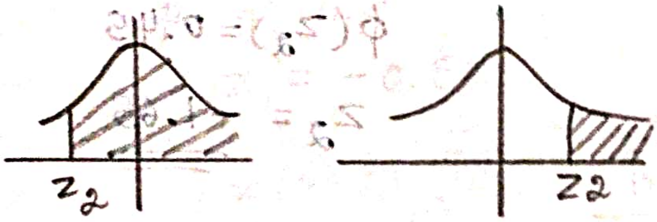




✗ $\because P(\text{this area})$ is > 0.5 (but given is 0.31)

✓

$P(z>z_2) = 0.08 \Rightarrow 0.5 - \phi(z_2) = 0.08$



$\phi(z_2) = 0.42$

$\therefore z_2 = 1.41$

$z_1 = \frac{45-\mu}{\sigma} = -0.5 \Rightarrow 45 = \mu - 0.5\sigma$

$z_2 = \frac{64-\mu}{\sigma} = 1.41 \Rightarrow 64 = \mu + 1.41\sigma$

$\mu - 0.5\sigma = 45$
$\mu + 1.41\sigma = 64$
$(-) \quad (-) \qquad (-)$

$-1.91\sigma = -19$

$\therefore \sigma = 9.947$

$\therefore \mu = 49.97$

4. Steel rods are manufactured to be 3 cms in diameter but they are acceptable if they are inside the limits 2.299 cms & 3.01 cms. It is observed that 5% are rejected as oversized & 5% are rejected as undersized. Assuming that the diameters are normally distributed, find the standard deviation of the distribution.

ans Given: $P(x>3.01) = 0.05$ & $P(x<2.299) = 0.05$

$\mu = ?$ & $\sigma = ?$

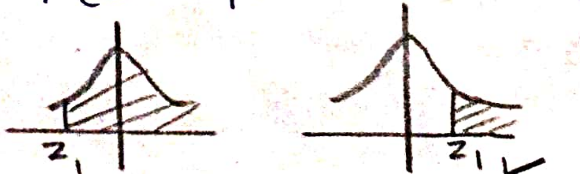
$z = \frac{x-3}{\sigma}$

$P(x>3.01) = 0.05$

$P(z>z_1) = 0.05$, $z_1 = \frac{3.01-\mu}{\sigma}$

$P(z<z_2) = 0.05$, $z_2 = \frac{2.299-\mu}{\sigma}$

$P(z>z_1) = 0.05 \Rightarrow 0.5 - \phi(z_1) = 0.05$

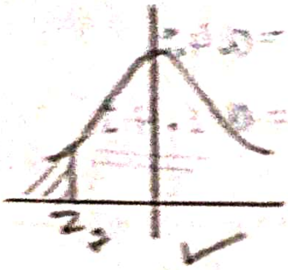


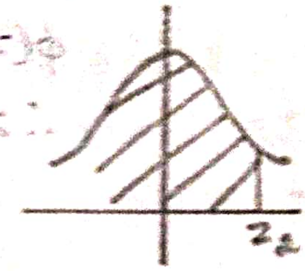
$\phi(z_1) = 0.45$

$\therefore z_1 = 1.65$

$P(Z<Z_2) = 0.05 \Rightarrow 0.5 - \phi(z_2) = 0.05$

$\phi(z_2) = 0.45$





$z_2 = \underline{\underline{-1.65}}$

$2.299 = \mu - 1.65\sigma$

$3.01 = \mu + 1.65\sigma$

$\Rightarrow \mu = 2.6545, \quad \sigma = 0.2154$

$\therefore SD(\sigma) = \underline{\underline{0.2154}}$

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5. In an examination taken by 500 candidates, the average & SD of marks obtained (normally distributed) are 40% & 10% respectively. Find approximately (i) how many will pass, if 50% is fixed as minimum?

(ii) What should be the minimum if 350 candidates are to pass?

(iii) How many have scored marks above 60%?

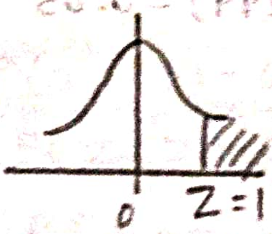
ans $x \to$ Marks of students in %

$\mu = 40$ & $\sigma = 10$, $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-40}{10}$

(i) $P(x \geq 50) = P(z \geq 1) = 0.5 - \phi(1)$ $\quad z = \dfrac{50-40}{10} = 1$

$= 0.5 - 0.3413$

$= 0.1587$



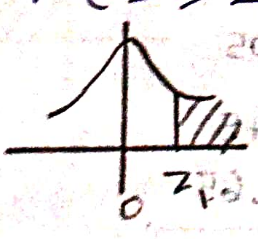
$\therefore$ No. of students $= 500 \times 0.1587 = 79.35$

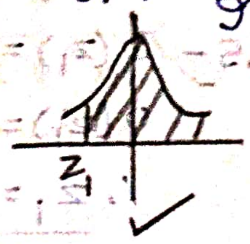
$\simeq \underline{\underline{79}}$

(ii) Let M be min. marks

$P(x \geq M) = \dfrac{350}{500} = 0.7$

$P(z \geq z_1) = 0.7$ & $z_1 = \dfrac{M-40}{10}$





$0.5 + \phi(z_1) = 0.7$

$\phi(z_1) = 0.2$

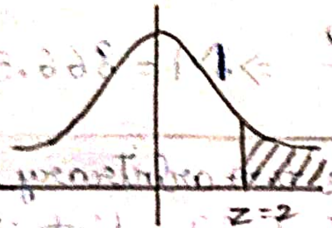
$z_1 = -0.52$

$z_1 = \frac{M-40}{10}$

$-0.52 = \frac{M-40}{10} \Rightarrow 40 - 5.2 = M \Rightarrow M = 34.8\%$

$M \simeq 35\%$

(iii) $P(x>60) = P(z>2) = 0.5 - \phi(2)$ $\qquad Z = \frac{60-40}{10} = 2$



$= 0.5 - 0.4772$

$= 0.0228$

$\therefore$ No. of students $= 500 \times 0.0228 = 11.4$

$\simeq 11$

6. The income of a group of 10,000 persons was found to be normally distributed with mean Rs 750 pm & SD of Rs 50. Show that, of this group, about 95% had income exceeding Rs. 668 & only 5% had income exceeding Rs. 832. Also find the lowest income among the richest 100.

ans $x \rightarrow$ Income of a person per month

$\mu = 750, \sigma = 50$; $z = \frac{x-\mu}{\sigma} = \frac{x-750}{50}$

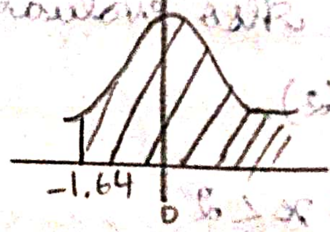
To prove that:

i) $P(x>668) = 0.95$ $\qquad z = \frac{668-750}{50}$

$P(z > -1.64) = 0.5 + \phi(1.64)$ $\qquad z = -1.64$

$= 0.5 + 0.4495$

$= 0.9495 \simeq 95\%$



No need

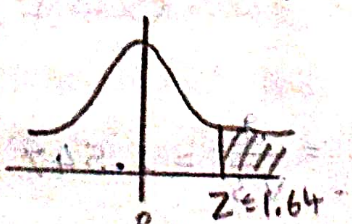
(No. of persons $= 10,000 \times 0.9495 = 9495$ which is 95% of 10000)

ii) $P(x>832) = 0.05$

$P(z>1.64) = 0.5 - \phi(1.64)$ $\qquad z = \frac{832-750}{50}$

$= 0.5 - 0.4495$ $\qquad z = 1.64$

$= 0.0505 \simeq 5\%$

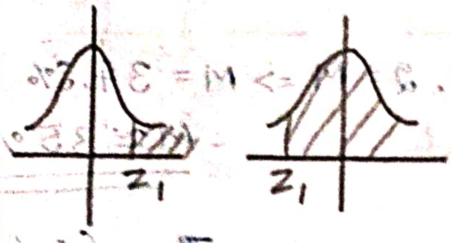


iii) Let M be the minimum income

1% of 10000 people is 100 [illegible]

$$P(x>M) = \frac{100}{10000} = 0.01$$

$$P(z>z_1) = 0.01 \; ; \; z_1 = \frac{M-750}{50}$$



$$P(z>z_1) = 0.01$$

$$0.5 - \phi(z_1) = 0.01$$

$$\phi(z_1) = 0.49$$

$$\underline{z_1 = 2.33}$$

$$2.33 = \frac{M-750}{50} \Rightarrow M = 866.5 \simeq \underline{867}$$

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☆ Uniform distribution: Let a & b be two arbitrary real constants such that $a<b$, then the probability distribution for which $u(a,b,x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$ is density function is called the uniform probability distribution in the interval (a,b); the corresponding continuous random variable x is called the uniform variate. Clearly $u(a,b,x) \geq 0$; $\int_{-\infty}^{\infty} u(a,b,x) = 1$.

→ Mean $= \frac{a+b}{2}$ & variance $= \frac{(b-a)^2}{12}$

→ $P(a<x<b) = 1$; $P(x \leq a) = P(x \geq b) = 0$.

☆ Problems:-

1. A random variable x is uniformly distributed for $-2<x<2$. Find the mean & SD. Also evaluate $P(x<1)$; $P(|x|>1)$; $P(|x-1| \geq 1/2)$.

ans $U(x) = \begin{cases} 1/4 \; \frac{1}{b-a} & -2<x<2 \\ 0 & \text{otherwise} \end{cases}$ $\qquad \underset{a}{-2} < x < \underset{b}{2}$

1) Mean $= \frac{a+b}{2} = 0$

$$S.D. = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{4^2}{12}} = \sqrt{\frac{16}{12}} = \frac{2}{\sqrt{3}} = 1.547$$

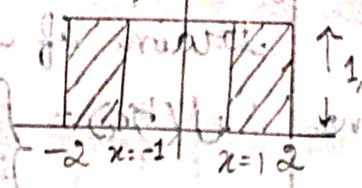
2) $P(x<1) = \int_{-2}^{1} u(x)\,dx = \int_{-2}^{1} \frac{1}{4}\,dx = \frac{1}{4}x\Big]_{-2}^{1} = \frac{3}{4}$

3) $P(|x|>1) = P(x>1 \text{ or } x<-1)$

$= P(x>1) + P(x<-1)$

$= 2P(x>1) = 2\int_{1}^{2} \frac{1}{4}\,dx = \frac{1}{2}(2-1)$

$= \frac{1}{2}$



4) $P(|x-1| \geq 1/2) = P(x-1 \geq 1/2 \text{ or } x-1 \leq -1/2)$

$= P(x \geq 3/2) + P(x \leq 1/2)$

$= \int_{3/2}^{2} \frac{1}{4}\,dx + \int_{-2}^{1/2} \frac{1}{4}\,dx$

$= \frac{1}{4}(2-3/2) + \frac{1}{4}(1/2+2)$

$= \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{5}{2}\right) = \frac{1}{8} + \frac{5}{8} = \frac{6}{8} = \frac{3}{4}$

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2. The daily amount of coffee (in litres) dispensed by a machine is uniformly distributed with $a=7$ & $b=10$. Determine the probability that the amount of coffee dispensed by the machine will be i) at most 8.8 litres (ii) more than 7.4 litres but less than 9.5 litres (iii) atleast 8.5 litres.

ans $U(x) = \begin{cases} \frac{1}{3} & 7<x<10 \\ 0 & \text{otherwise} \end{cases}$

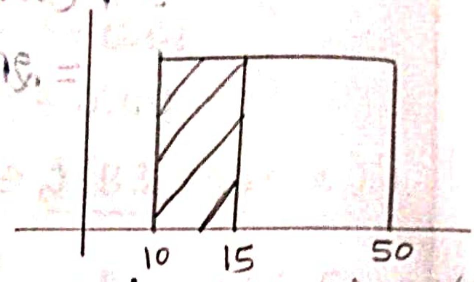
i) $P(x<8.8) = \int_{7}^{8.8} \frac{1}{3}\,dx = \frac{1.8}{3} = 0.6$

ii) $P(7.4<x<9.5) = \int_{7.4}^{9.5} \frac{1}{3}\,dx = \frac{9.5-7.4}{3} = 0.7$

iii) $P(x>8.5) = \int_{8.5}^{10} \frac{1}{3}\,dx = \frac{10-8.5}{3} = \frac{1.5}{3} = 0.5$

3. The driving time X from a person's home to the train station is uniformly distributed as $U[10,50]$. If it takes 2 minutes to board the train, determine the probability that the person catches the 7am train if he starts at 6:43 am from his home.

ans $U(x) = \begin{cases} 1/40 & 10<x<50 \\ 0 & \text{otherwise} \end{cases}$



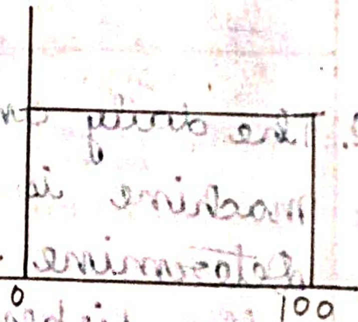
$P(10< x <15) = \int_{10}^{15} \frac{1}{40} dx = \frac{15-10}{40}$

$= \frac{5}{40} = \frac{1}{8} = \underline{0.125}$

Min (10 min drive) so 6:53 — 58 max, then can board
10min 15min

4. The amount charged for a visit to a dental clinic is uniformly distributed from 0 to 1000 (in INR). Given that the amount charged for a visit exceeds Rs. 500; calculate the probability that is exceeds Rs. 750.

ans $U(x) = \begin{cases} \frac{1}{1000} & 0<x<1000 \\ 0 & \text{otherwise} \end{cases}$



Conditional Probability
Let A be $x>750$ & B be $x\geq 500$.
~~Probability of A given~~

$P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{P(x\geq 750 \cap x\geq 500)}{P(x\geq 500)}$

$= \frac{P(x>750)}{P(x>500)}$

$= \frac{\int_{750}^{1000} \frac{1}{1000} dx}{\int_{500}^{1000} \frac{1}{1000} dx} = \frac{1000-750}{1000-500} = \frac{250}{500} = \underline{\frac{1}{2}}$

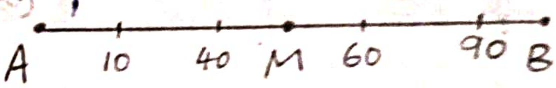
5. A bus travels between two cities A & B which are 100 miles apart. If the bus has a breakdown, the

distance X of the point of breakdown from the city A has a uniform distribution $U[0, 100]$. There are service garages in the city A, city B & midway between the two cities such that in case of a breakdown a tow truck is sent from the garage nearest to the point of breakdown.

i) What is the probability that the tow truck has to travel more than 10 miles to reach the bus?

ii) Would it be more 'efficient' if the 3 service garages were placed at 25, 50 & 75 miles from city A.

ans $U(x) = \begin{cases} \frac{1}{100} & 0 < x < 100 \\ 0 & \text{otherwise} \end{cases}$



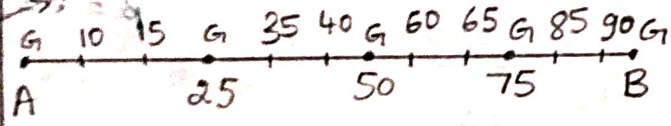
i) $P(10 < x < 40 \text{ or } 60 < x < 90) = P(10 < x < 40) + P(60 < x < 90)$

$$\Rightarrow \int_{10}^{40} \frac{1}{100}\,dx + \int_{60}^{90} \frac{1}{100}\,dx$$

$$= \frac{30}{100} + \frac{30}{100} = \frac{60}{100} = \underline{\underline{0.6}}$$

At A & B garages are there

ii)



$P(10 < x < 15) + P(35 < x < 40) + P(60 < x < 65) + P(85 < x < 90)$

$$\int_{10}^{15} \frac{1}{100}\,dx + \int_{35}^{40} \frac{1}{100}\,dx + \int_{60}^{65} \frac{1}{100}\,dx + \int_{85}^{90} \frac{1}{100}\,dx =$$

$$\frac{5}{100} + \frac{5}{100} + \frac{5}{100} + \frac{5}{100} = \frac{20}{100} = \frac{1}{5} = \underline{\underline{0.2}}$$

$\therefore$ ii) is more efficient

☆ Gamma Distribution: Let α & β be two arbitrary constants both of which are positive. Also, let $\Gamma(\alpha)$ be the Gamma function. Then the probability distribution for which $\gamma(\alpha, \beta, x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}, & x \geq 0, \\ 0 & \text{otherwise} \end{cases}$

is the density function, is called the gamma distribution & the corresponding continuous random variable x is called the gamma variate. The constants α & β present in the density function $\gamma(\alpha, \beta, x)$ are called the parameters of the distribution.

Clearly $\gamma(\alpha, \beta, x) \geq 0$ & $\int_{-\infty}^{\infty} \gamma(\alpha, \beta, x)\,dx = 1$

★ Mean & Variance of Gamma distribution

→ Mean $= \alpha\beta$ & Variance $= \alpha\beta^2$

NOTE: $P(a \leq x \leq b) = \frac{1}{\Gamma(\alpha)} \int_{a/\beta}^{b/\beta} t^{\alpha-1} e^{-t}\,dt$, $t = x/\beta$

1. The demand for certain item is distributed as Gamma distribution with mean 8 & variance 32. Find the probability that there will be a demand for at least 10 items.

ans Mean = 8 & variance = 32 $\alpha\beta = 8$, $\alpha\beta^2 = 32$

$X \rightarrow$ demand for an item $\alpha\beta\beta = 32$

$\beta = 32/8 = 4$

$\alpha = 2$, $\beta = 4$

$\gamma(\alpha, \beta, x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma\alpha}$

$\gamma(2, 4, x) = \frac{x e^{-x/4}}{4^2 \Gamma 2} = \frac{x e^{-x/4}}{16}$

$P(x > 10) = \int_{10}^{\infty} \frac{x e^{-x/4}}{16}\,dx = \frac{1}{16}\left[-4x e^{-x/4} - 16 e^{-x/4}\right]_{10}^{\infty}$

$= \frac{1}{16}\left[4(10) e^{-10/4} + 16 e^{-10/4}\right]$

$= \frac{1}{16}[3.283 + 1.313] = 0.287$

2. The no. of road accidents per day in a certain city is distributed as a gamma variate with an average of 6 & variance 18. Find the probability that there will be (i) more than 8 accidents

(ii) between 5 & 8

ans $x \to$ no. of accidents per day

mean = 6 & Variance = 18 $\quad 6(\beta) = 18;\ \beta = 3;\ \alpha = 2$

$\alpha\beta = 6 \qquad \alpha\beta^2 = 18$

$$\gamma(\alpha, \beta, x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma\alpha}$$

$$\gamma(2, 3, x) = \frac{x e^{-x/3}}{3^2 \Gamma(2)} = \frac{x e^{-x/3}}{9\,(1)}$$

$x > 0$

$$\gamma(2, 3; x) = \frac{x e^{-x/3}}{9} \qquad \frac{x e^{-x/3}}{-1/3} - \frac{e^{-x/3}}{1/9}$$

i) $P(x > 8) = \int_8^\infty \frac{x e^{-x/3}}{9} dx = \frac{1}{9}\left[-3xe^{-x/3} - 9e^{-x/3}\right]_8^\infty$ Bernoulli

$= -\frac{1}{9}\left[-3(8)e^{-8/3} - 9e^{-8/3}\right]$

$= +\frac{1}{9}(33e^{-8/3}) = \frac{2.2929}{9} = 0.2547$

ii) $P(5 < x < 8) = \int_5^8 \frac{xe^{-x/3}}{9} dx = \frac{1}{9}\left[-3xe^{-x/3} - 9e^{-x/3}\right]_5^8$

$= \frac{1}{9}\left[-24e^{-8/3} - 9e^{-8/3}\right] - \frac{1}{9}\left[-15e^{-5/3} - 9e^{-5/3}\right]$

$= \frac{-33e^{-8/3}}{9} + \frac{24}{9}e^{-5/3} = 0.248$

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3. After the appointment of a new sales manager in a shop, the increase in sales per day in the shop is found as a gamma variate with parameters $\alpha = 2$ & $\beta = 2000$. What is the probability that the increase in salestax returns on a randomly chosen day exceeds Rs 200, given that the sales tax is 5% of the sales.

ans $\alpha = 2,\ \beta = 2000$

$$\gamma(\alpha, \beta, x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\Gamma\alpha} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma(2; 2000, x) = \begin{cases} \frac{x^1 e^{-x/2000}}{(2000)^2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let x be sales per day

Sales tax 5% of X

$P\left(\frac{5}{100} \times x > 200\right)$

$P(x > 4000)$

$$= \int_{4000}^{\infty} \gamma(\alpha, \beta, x)\, dx = \int_{4000}^{\infty} \frac{x e^{-x/2000}}{(2000)^2}\, dx$$

$$= \frac{1}{(2000)^2}\left[\frac{x e^{-x/2000}}{-1/2000} - \frac{e^{-x/2000}}{(-1/2000)^2}\right]_{4000}^{\infty}$$

$$= \frac{-1}{(2000)^2}\left[-(4000) e^{-2}(2000) + (2000)^2 e^{-2}\right]$$

$$= \underline{\underline{0.1353}}$$

4. The daily sales of a certain brand of bicycles in a city in excess of (more than) 1000 pieces is distributed as the gamma distribution with the parameters $\alpha = 2$ & $\beta = 500$. The city has a daily stock of 1500 pieces of the brand. Find the probability that the stock is insufficient on a particular day.

ans $\alpha = 2$, $\beta = 500$

$$\gamma(\alpha, \beta, x) = \begin{cases} \dfrac{x^1 e^{-x/500}}{500^2 \Gamma 2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let Y be sales per day

X = Y − 1000 is a gamma variate

$P(Y > 1500) = P(X > 500)$

$$= \int_{500}^{\infty} \frac{x e^{-x/500}}{(500)^2}\, dx = \frac{1}{(500)^2}\int_{500}^{\infty} x e^{-x/500}$$

$$= \frac{1}{(500)^2}\left[\frac{x e^{-x/500}}{-1/500} - \frac{e^{-x/500}}{(1/500)^2}\right]_{500}^{\infty}$$

$$= \frac{-1}{(500)^2}\left[-500\,(500)e^{-1} - (500)^2 e^{-1}\right]$$

$$= e^{-1} + e^{-1} = 2e^{-1} = 0.7357$$