B.M.S. COLLEGE OF ENGINEERING, BENGALURU-19
Autonomous Institute, Affiliated to VTU
DEPARTMENT OF MATHEMATICS
THIRD SEMESTER B.E COURSE(CSE/ISE/CSE-AIDS/CS-IOT/CSE-DS)

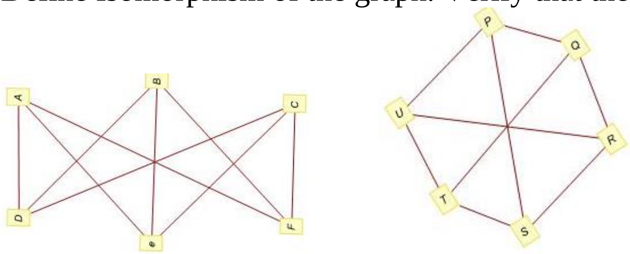
Course Title: Statistics and Discrete Mathematics **Course Code 23MA3BSSDM**

UNIT-1 Graph Theory

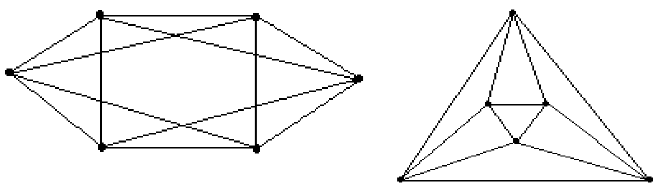
1. Define the following with an example.
   (i) Digraph (ii) complete bipartite graph (iii) 3-D hypercube
2. Suppose that in a group of 5 people: A, B, C, D, and E, the following pairs of people are acquainted with each other- A and C, A and D, B and C, C and D & C and E.
   (a) Draw a graph G to represent this situation. (b) Identity the degree of each vertex.
3. Model the following situation as a graph and write its vertex set and the edge set. Also, write the degree of each vertex. In Netherland, there is a highway from Amsterdam to Breda, another highway from Amsterdam to Cappele aan den IJssel, a highway from Breda to Dordrecht, a highway from Breda to Ede and another one from Dordrecht to Ede, and a highway from Cappele aan den IJssel to Ede.
4. Let P,Q, R,S and T represent 5 cricket teams. Suppose that the teams P,Q,R have played one game with each other and the teams P,S,T have played one game with each other. Represent this situation as a graph and hence determine:
   (i) The teams that have not played with each other.
   (ii) The number of games played by each team.
5. For a graph $G=(V,E)$, what is the largest possible value of $|V|$ if $|E|=35$ and $\deg(v)\geq 3, \forall v\in V$?
6. For a graph $G=(V,E)$, what is the largest possible value of $|V|$ if $|E|=19$ and $\deg(v)\geq 4, \forall v\in V$?
7. For a graph with n vertices and m edges if δ is the minimum and Δ is the maximum of the degrees of the vertices, show that $\delta \leq \frac{2m}{n} \leq \Delta$.
8. Determine the order of the graph in the following cases.
   (i) G is a cubic graph with 9 edges.
   (ii) G has 10 edges with 2 vertices of degree 4 and all other vertices of degree 3.
   (iii) G is a regular graph with 15 edges.

9. Let G be a graph of order 9 such that each vertex has degree 5 or 6. Prove that at least 5 vertices have degree 6 or at least 6 vertices have degree 5.
10. Show that every simple graph order greater than or equal to 2 must have atleast 2 vertices of the same degree.
11. Define Isomorphism of graphs. Verify that the given two graphs are isomorphic or not11

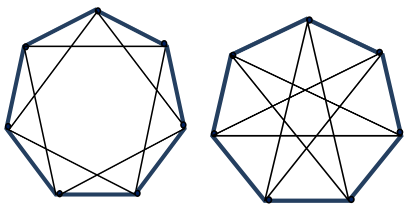
12. Define Isomorphism of the graph. Verify that the given two graphs are isomorphic or not.



13. Verify whether the following graphs are isomorphic or not.



14. With proper labeling show that the following graphs are isomorphic.



15. Suppose a committee has seven members, these members meet each day for lunch at a round table. They decide to sit in such a way that every member has different neighbors at each lunch. How many ways can this arrangement last?
16. Suppose that a tree T has two vertices of degree 2, four vertices of degree 3 and three vertices of degree 4. Find the number of leaves in T.
17. Let G be a simple graph with n vertices and m edges where m is at least 3. If $m \geq \frac{(n-1)(n-2)}{2} + 2$, prove that G is Hamiltonian graph.
18. Prove that a connected graph G remains connected even after removing an edge *e* from G if and only if *e* is a part of some cycle in G.

19. Let G be a disconnected graph of even order n with two components each of which is complete. Prove that G has a minimum of $\frac{n(n-2)}{4}$ edges.

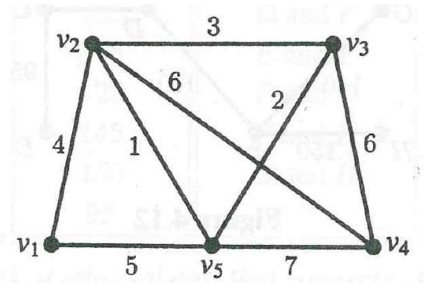
20. If G is a simple graph with n vertices in which the degree of every vertex is at least $(n-1)/2$, prove that G is connected.

21. Show that a connected graph with exactly 2 vertices of odd degree has an Euler trail.

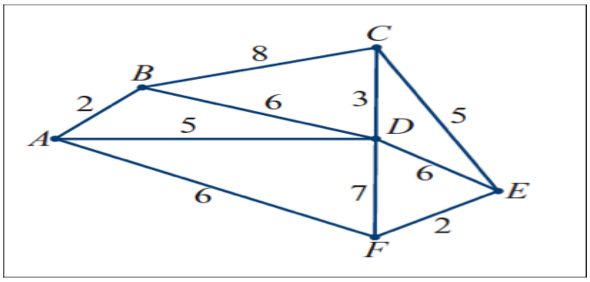
22. Find all the spanning trees of the given graph:

23. For the given graph, draw the following:

(i) Spanning Tree (ii) Edge Disjoint Subgraph (iii)Induced Subgraph

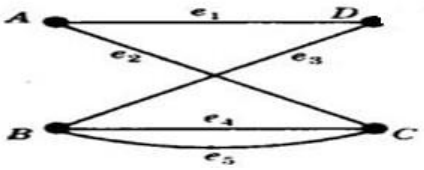


24. For the given graph, draw the following:(i) Spanning Tree (ii) Edge Disjoint Subgraph (iii)Induced Subgraph

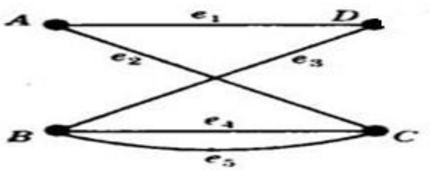


Matrix Representation of a graph.

1. For the given graph, write its incidence matrix. Hence, write any three observations on it.



2. For the given graph, write its adjacency matrix. Hence, write any three observations on it.



3. Draw the graph G whose incidence matrix is given and hence obtain the adjacency matrix of the corresponding graph G

$$A(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. For the given adjacency matrix $A(G)$, construct incidence matrix and also write any three observations on incidence matrix.

$$A(G) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

5. Draw the graph G whose incidence matrix is given and hence obtain the adjacency matrix of the corresponding graph G.

$$A(G) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

6. Obtain the incidence matrix for the graph whose adjacency matrix is given below.

$$X(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

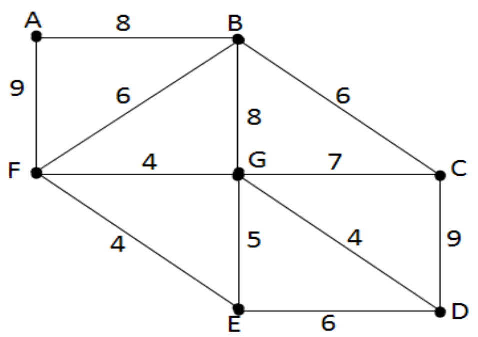
Kruskal's algorithm

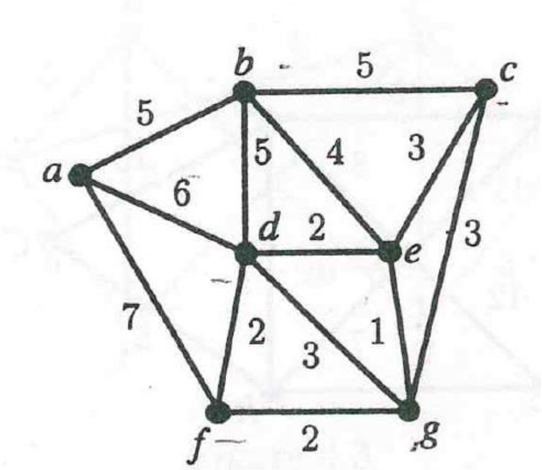
1. Eight cities A,B,C,D,E,F,G,H are required to be connected by a new railway network. The possible tracks and the cost involved to lay them (in crores of rupees) are summarized in the Table:

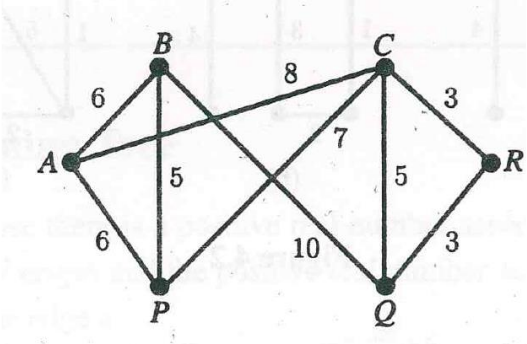
Track between	Cost	Track between	Cost
		C and E	95
A and B	155	D and F	100
A and D	145	E and F	150
A and G	120	F and G	140
B and C	145	F and H	150
C and D	150	G and H	160

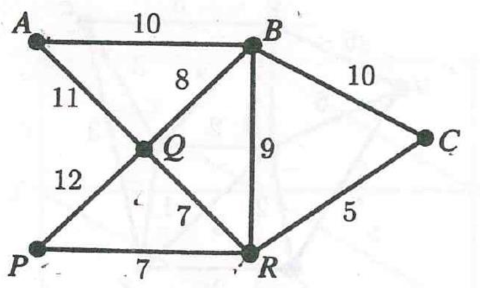
Determine a railway network of minimal cost that connects all these cities using Kruskal's algorithm.

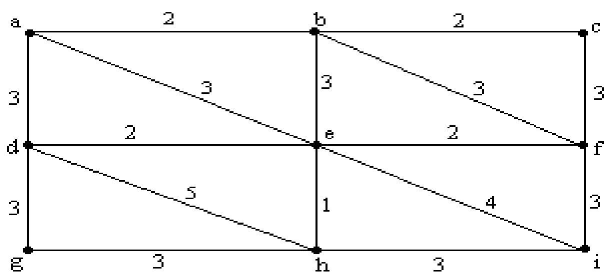
2. Apply Kruskal's algorithm to find a minimal spanning tree for the following weighted graphs and hence find its weight.

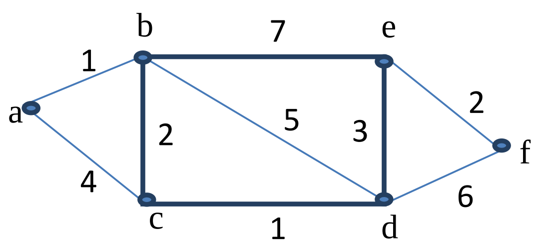


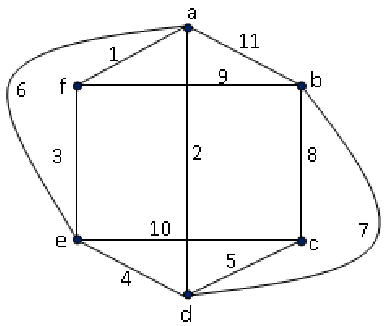


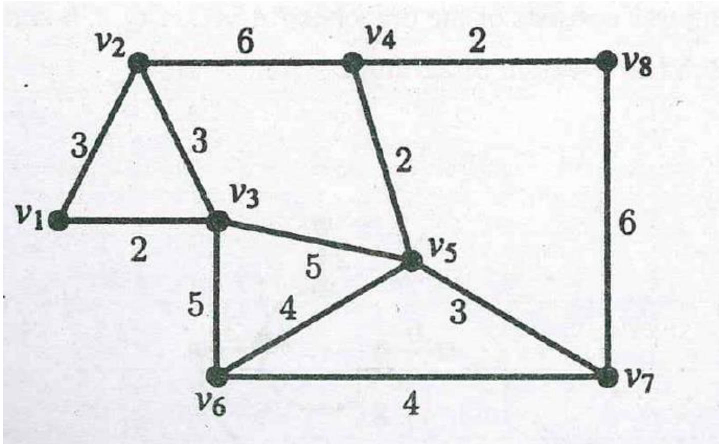


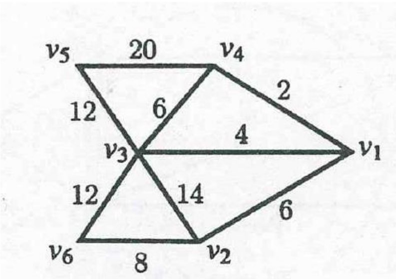


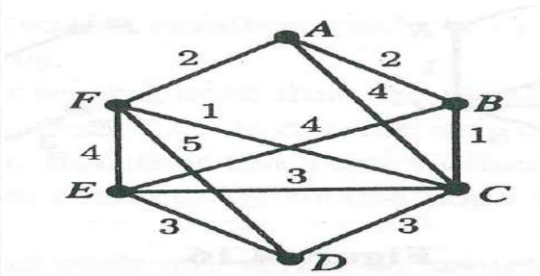





a
b
c
d
e
f
1
11
9
6
3
2
8
10
7
5
4


v2
6
v4
2
v8
3
3
2
v1
v3
2
6
5
v5
5
4
3
v6
v7
4


v5
20
v4
2
12
6
v3
4
v1
12
14
6
v6
8
v2


A
2
2
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F
B
1
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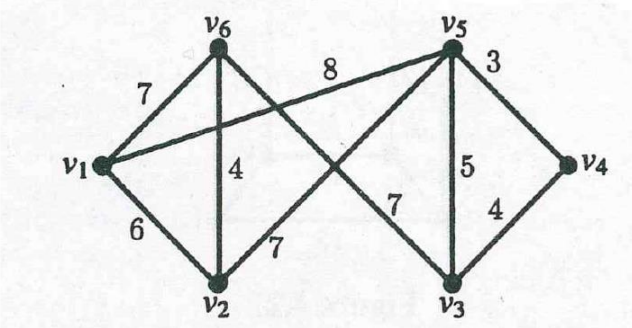
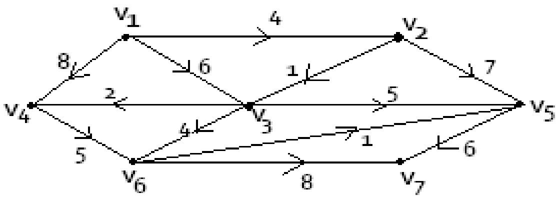
5
1
4
3
E
C
3
3
D

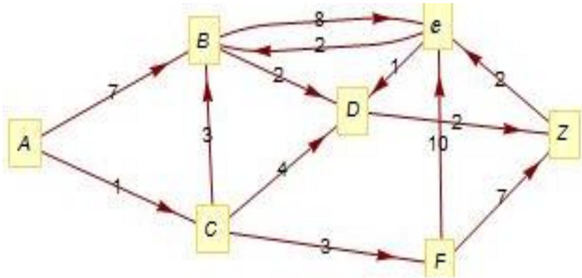

v6
v5
3
8
7
v1
4
5
v4
7
4
6
7
v2
v3

Dijkstra's

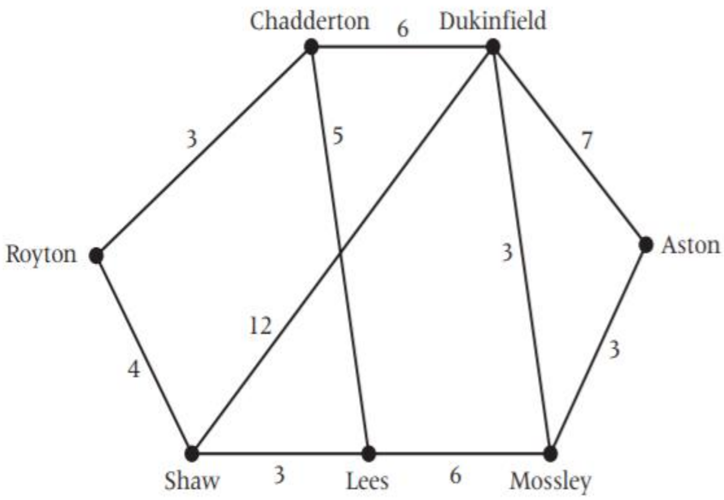
1. Apply Dijkstra's algorithm to find the shortest path and its weight from vertex v_1 to vertex v_5 from the weighted, directed network shown below.



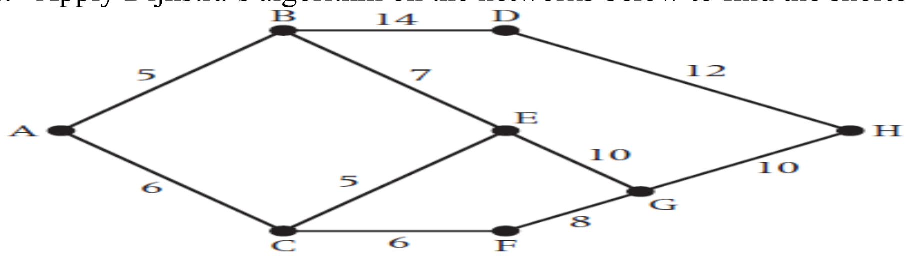
2. Apply Dijkstra's algorithm to find the shortest path and its weight from vertex A to vertex Z from the weighted, directed network



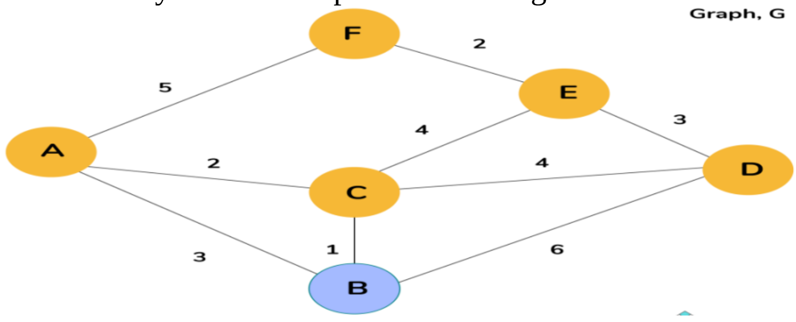
3. The diagram below shows roads connecting villages near to Royton. The numbers on each arc represent the distance, in miles, along each road. Leon lives in Royton and works in Ashton. Use Dijkstra's algorithm to find the minimum distance for Leon's journey to work.



4. Apply Dijkstra's algorithm on the networks below to find the shortest distance from A to H.



5. Consider the map below. The cities have been selected and marked from alphabets A to F and every edge has a cost associated with it. We need to travel from Bengaluru (Vertex B) to all other places. Identify the shortest paths from Bengaluru to other destinations.



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UNIT-1

GRAPH THEORY

★ The 4 Colour Theorem:

→ Conjecture proposed in 1852 by Mobius-Given

→ Every planar graph is four-colourable (Thomas in 1998; Wilson in 2014).

★ Graph: A graph is a pair (V, E), where V is a non-empty set & E is a set of <u>unordered</u> pair of elements taken from set V. Here elements of set V are called vertices & elements of set E are called edges. V is called the vertex set & set E is called the edge set. The graph (V, E) is denoted by G.

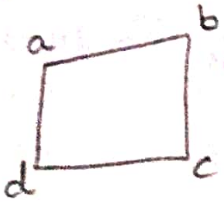
→ V → set of all vertices

→ E → edge set

{(a, b) & (b, a) are same then → unordered pair; (a, b) is different from (b, a) → ordered pair}

★ Graph Theory → Study of relation between dots & lines.

Eg:- G:



$V = \{a, b, c, d\}$

$E = \{ab, bc, cd, ad\}$

NOTE: 1. According to the definition of graph G, the vertex set should be non-empty.

2. Edge set can be empty

★ Type of Graphs:-

1. Null graph: A graph with no edges is called null graph. (Or) A graph in which the edge set is empty is a null graph.

Eg:- G_1:

G_2:

G_3:

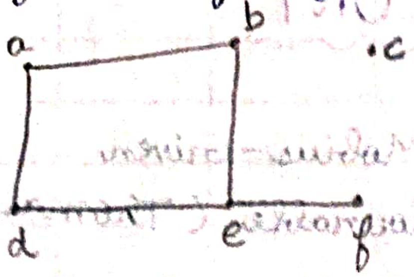
NOTE: Trivial Graph: A null graph with only one vertex is a trivial graph.

2. Finite Graph: A graph with only a finite number of vertices as well as finite no. of edges is called finite graph otherwise infinite graph.

NOTE: 1. Order: Order is the no. of vertices in G.

2. Size : No. of edges in G.

Eg:- G:

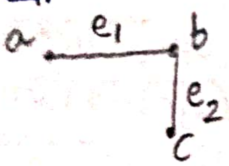


order : 6
Size : 5

3. Vertices of graph G are denoted by $v_1, v_2, \ldots, v_n$ & edges are denoted by $e_1, e_2, \ldots, e_m$.

4. If v_i & v_j denote two vertices of graph & e is an edge joining v_i & v_j, then v_i & v_j are called end vertices of e.

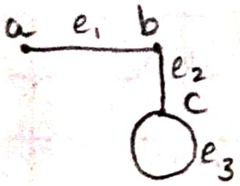
Eg:-



End vertices of e_1 are a, b.
b, c are end vertices of e_2.

5. Self loop: An edge whose end vertices co-incide (or) an edge joining a vertex to itself is called a self loop.

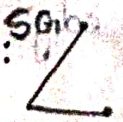
Eg:- G:



Here, e_3 is a self loop.

6. Parallel edges or multiple edges: Two or more edges with same pair of end vertices are called multiple edges.

3. Simple graph: A graph that does not contain loops & multiple edges is called a simple graph.

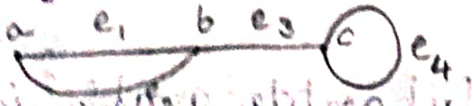
Eg:- G_1: SG  G_2: SG  G_3: SG  G_4: SG  G_5: 

G_6:  G_7:  G_8: SG 

4. Labeled graph: A graph in which vertices are not labeled is called unlabeled graph else labeled graph.

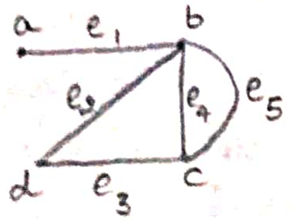
NOTE: 1. When a vertex v of a graph is an end vertex of an edge e of a graph G, we say the edge e is incident on vertex v. Since every edge has two end vertices, every edge is incident on two vertices, one on each end (except self loop).

Eg:-



2. Adjacency: Two non-parallel edges are said to be adjacent edges if they are incident on a common vertex. Two vertices are said to be adjacent vertices if there is an edge joining them.

Eg:- G:



Adjacent edges: ① e_1, e_2, e_4, e_5
② e_2, e_3
③ e_3, e_4, e_5

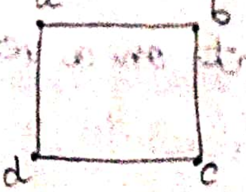
Adjacent vertices: ab, bc, bd, cd

5. Complete graph: A simple graph of order $n \geq 2$ in which there is an edge between every pair of vertices is called a complete graph & is denoted by K_n.

Eg:- K_2: K_3:  K_4:  K_5:  K_6: 

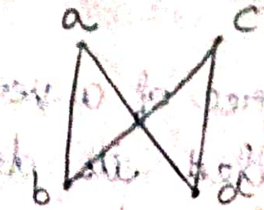
6. Bipartite graph: A simple graph G is such that its vertex set V is the union of two of its mutually disjoint non-empty subsets V_1 & V_2 which are such that each edge in G joins a vertex in V_1 & a vertex in V_2. Then G is called a bipartite graph denoted by $G(V_1, V_2, E)$, where E is the edge set & V_1 and V_2 are bipartites.

Eg:-(i) G:

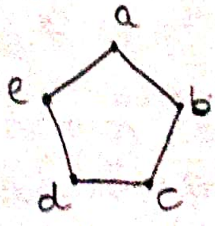


$V = \{a, b, c, d\}$
$V_1 = \{a, c\}$
$V_2 = \{b, d\}$

G:



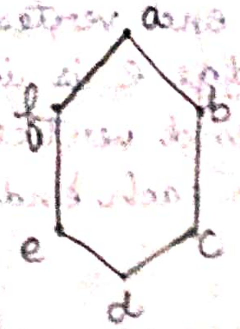
(ii)



$V = \{a, b, c, d, e\}$
$V_1 = \{a, c$
$V_2 = \{b, e$

This is not a bipartite graph.
{∵ d cannot be anywhere}

(iii) G:

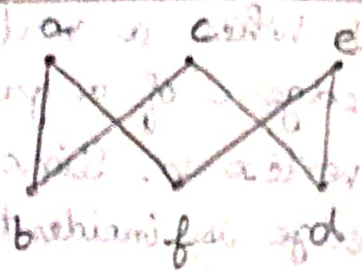


$V = \{a, b, c, d, e, f\}$

$V_1 = \{a, c, e\}$

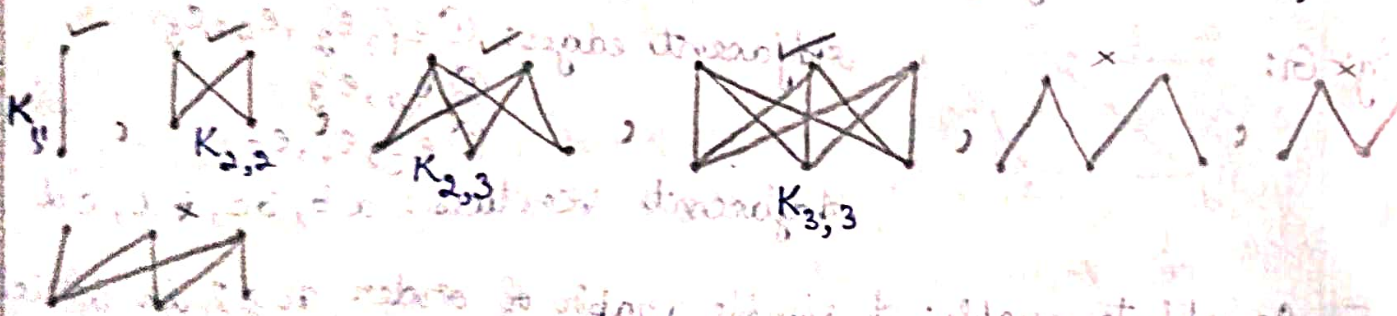
$V_2 = \{b, f, d\}$

G:



7. Complete bipartite graph: A bipartite graph $G(V_1, V_2, E)$ in which there is an edge between every vertex in V_1 & every vertex in V_2 is called a complete bipartite graph.
A complete graph $G(V_1, V_2, E)$ in which the bipartite V_1 & V_2 contains r & s vertices respectively with $r \leq s$ denoted by $K_{r,s}$.

Eg:-



NOTE:- (i) $K_{r,s}$ has $r+s$ vertices & rs edges.
(ii) $K_{3,3}$ is called Kuratowski's second graph.

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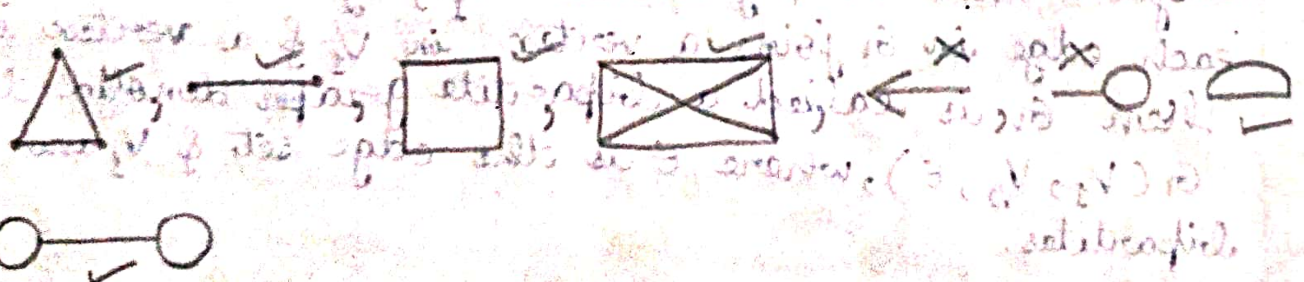
8. Regular Graph: A graph in which all the vertices are of the same degree k is called a regular graph of degree k or k-regular graph.

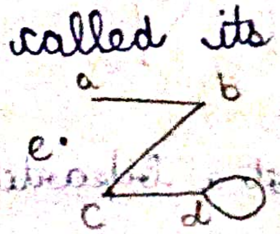
Eg:-



NOTE: Degree of a vertex: No. of edges incident on a vertex is called its degree.

Any loop contributes 2 to the degree.

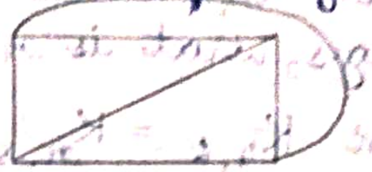
Eg:-



deg(a) = 1
deg(b) = 2
deg(c) = 2
deg(d) = 3
deg(e) = 0

★ Problems:-

1. Which of the following graph is complete graph?

G_1:  G_2: 

ans G_1 is not a complete graph as it is not a simple graph. G_2 is a simple graph where every vertex is connected to remaining vertices of G_2. $\therefore$ G_2 is a complete graph.

2. Show that a complete graph with n vertices, k_n has $\frac{n(n-1)}{2}$ edges.

ans Proof: k_n has n vertices. No. of edges in k_n is equal to choosing any two vertices out of n vertices & making them adjacent which is equal to n_{C_2}

$$n_{C_2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2} = \frac{n(n-1)}{2}$$

$\therefore$ No. of edges in K_n is $\frac{n(n-1)}{2}$

3. Show that a simple graph of order 4 & size 7 & a complete graph of order 4 & size 5 do not exist.

ans No. of edges in a simple graph $\leq$ No. of edges in a complete graph 

Given $n = 4$

No. of edges in a simple graph with $n = 4$ is $\leq \frac{4(3)}{2}$

≤ 6

Given $m = 7$

$\therefore$ A simple graph of order 4 & size 7 does not exist.

A complete graph has $\frac{n(n-1)}{2}$ edges.

Given $n = 4$, no. of edges in complete graph with $n = 4$ is 6 $\left(\because \frac{4(4-1)}{2} = 6\right)$. 

Given $n = 4$ & $m = 5$, but this is a contradiction.

$\therefore$ No complete graph exists with $n = 4$ & $m = 5$.

4. How many edges & vertices are there in the complete bipartite graphs $K_{4,6}$. If graph $K_{r,12}$ has 72 edges, what is r?

ans Given $K_{4,6} = K_{r,s}$

No. of vertices in $K_{4,6} = r+s = 4+6 = \underline{\underline{10}}$

No. of edges in $K_{4,6} = rs = 4\times 6 = \underline{24}$

Given $K_{r,12}$ has 72 edges, i.e. $rs = 12$

WKT, No. of edges in $K_{r,12} = r\times 12$

$72 = r\times 12$

$\therefore \underline{\underline{r = 6}}$

9/10/24

☆ Hand Shaking Property [First theorem of graph theory]:-

The sum of the degrees of all the vertices in a graph is an even number & this number is equal to twice the number of edges in the graph.

$$\sum_{i=1}^{n} d(V_i) = 2m$$

1. At a party, each person shakes hands with every other person exactly once. If there were a total of 28 handshakes, how many people attended the party?

ans From handshaking property, $\sum_{i=1}^{n} d(V_i) = 2m$

Given: $m = 28$

$n = ?$

$d(V_i) = n-1$

$$\sum_{i=1}^{n} (n-1) = 2(28)$$

~~2+2+...+2 = 56~~ $\underbrace{(n-1)+(n-1)+\dots(n-1)}_{n} = 56$

$n(n-1) = 56 \Rightarrow n^2-n-56=0 \Rightarrow n^2-8n+7n-56=0$

$\Rightarrow n(n-8)+7(n-8) \Rightarrow \underline{n=8} \text{ or } n=-7$. $\therefore$ 8 people attended the party.

2. In a social network, if the sum of degrees of all users is 90, how many connections (edges) are there in the network?

ans Given : $\sum_{i=1}^{n} d(v_i) = 90$

WKT, $\sum_{i=1}^{n} d(v_i) = 2m$

$90 = 2m$

$\therefore \underline{m = 45}$

3. In every graph, the number of vertices of odd degrees is even.

ans Let G be a graph with n vertices & m edges. Let $v_1, v_2, \ldots, v_n$ be n vertices.

Let $v_1, v_2, \ldots, v_k$ be vertices with even degree & remaining $v_{k+1}, \ldots, v_n$ be vertices of odd degree.

From Hand Shaking Property, $\sum_{i=1}^{n} d(v_i) = 2m$

$$\sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{n} d(v_i) = 2m \quad \text{—} ①$$

The first sum on LHS of ① is an even number (sum of even numbers is even)

$$\sum_{i=k+1}^{n} d(v_i) = 2m - \sum_{i=1}^{k} d(v_i) \quad \text{—} ②$$

RHS of eq^n ② is an even number (even − even = even)

$$\sum_{i=k+1}^{n} d(v_i) = \text{Even} \quad \text{—} ③$$

(∴ Sum of odd degree vertices in G is even.)

∴ Sum of odd degree vertices on LHS of ③.

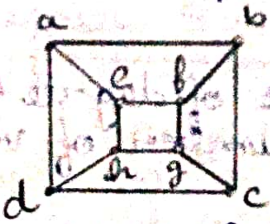
∴ There has to be even no. of terms on LHS of ③

(odd numbers should be added even times to make sum even)

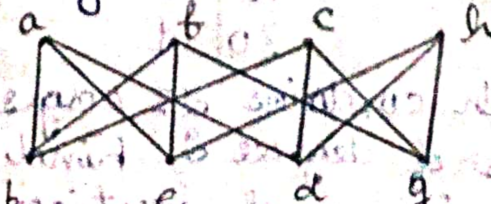
∴ In any graph G, number of vertices of odd degrees is even.

4. Show that the hypercube Q_3 is a bipartite graph which is not a complete bipartite graph.

ans Q_3:



G:



$V_1 = \{a, f, c, h\}$

$V_2 = \{b, e, d, g\}$

As $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$ & every edge of Q_3 joins a vertex of V_1 to a vertex of V_2.

$\therefore Q_3$ is a bipartite graph

Q_3 is not complete bipartite as vertices of V_1 are not adjacent to all vertices of V_2. [Eg:- a & g are not adjacent]

5. Prove that the k-dimensional hypercube Q_k has $k2^{k-1}$ edges. determine the number of edges of Q_8.

ans No. of vertices in $Q_k = 2^k$ & degree of every vertex in Q_k is k.

By handshaking property, $\sum_{i=1}^{2^k} d(v_i) = 2m$

$$\underbrace{k+k+\dots+k}_{2^k \text{ times}} = 2m$$

$$2^k k = 2m$$

$$m = k2^{k-1}$$

No. of edges in $Q_8 = 8.2^7 = 2^{10}$

6. Find number of vertices of graph G which is regular graph with 15 edges.

ans Let degree of every vertex in G be k.

By handshaking property: $\sum_{i=1}^{n} d(v_i) = 2m$

$$\underbrace{k+k+\dots+k}_{n \text{ times}} = 2m$$

$$nk = 2m = 30$$

$$n = 30/k$$

k	n
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

7. A graph contains 21 edges, 3 vertices of degree 4 & all other vertices of degree 2. Find total number of vertices.

ans Let n be no. of vertices in G

$m = 21$, n: 3 (deg 4), $n-3$ (deg 2)

By HSP, $\sum_{i=1}^{n} d(v_i) = 2m$

$$\underbrace{4+4+4+2+2+\dots+2}_{(n-3)\text{ times}} = 2(21)$$

$$12 + 2(n-3) = 42$$

$$2n - 6 = 30$$

$$2n = 36$$

$$\therefore n = 18$$

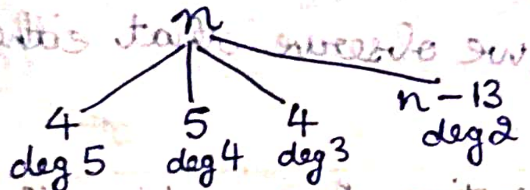
$\therefore$ Total number of vertices is 18.

---

8. A simple graph contains 35 edges, four vertices of degree 5, five vertices of degree 4 & four vertices of degree 3. Find the number of vertices with degree 2.

ans Let n be no. of vertices in G

$$m = 35$$



By HSP, $\sum_{i=1}^{n} d(v_i) = 2m$

$$5+5+5+5+4+4+4+4+4+3+3+3+3+\underbrace{2+2+\dots+2}_{(n-13)\text{ times}} = 2(35)$$

$$5(4) + 4(5) + 3(4) + 2(n-13) = 70$$

$$20 + 20 + 12 + 2n - 26 = 70$$

$$52 - 26 + 2n = 70$$

$$26 + 2n = 70$$

$$2n = 44$$

$$\therefore n = 22$$

$\therefore$ Total no. of vertices = 22.

No. of vertices with deg 2 = 22 − 13
= 9

---

9. Let G be a graph of order 9 such that each vertex has degree 5 or 6. Prove that at least 5 vertices have degree 6 or atleast 6 vertices of degree 5.

ans Let G be a graph with 9 vertices & m edges. Out of 9 vertices, let p vertices be of deg 5 & remaining $9-p=q$ vertices be of deg 6.

To prove: $p \geq 6$ or $q \geq 5$

By HSP: $\sum_{i=1}^{9} d(v_i) = 2m$

$$\underbrace{5+\dots+5}_{p \text{ times}} + \underbrace{6+\dots+6}_{q \text{ times}} = 2m$$

$$5p + 6q = 2m$$
$$5p + 6(9-p) = 2m$$
$$54 - p = 2m$$
$$p = 54 - 2m$$
$$p \to \text{even}$$

p	$q = 9-p$
0	9
2	7
4	5
6	3
8	1

In the above cases, we observe that either $p \geq 6$ or $q \geq 5$.

10. For a graph with n verities & m edges, if δ is the minimum & Δ is the maximum of degrees, show that $\delta \leq \frac{2m}{n} \leq \Delta$.

ans Let G be a graph with n verities & m edges. Let $d_1, d_2, \dots, d_n$ be degree of verities $v_1, v_2, \dots, v_n$. Let δ be minimum & Δ be maximum degree among $d_1, d_2, \dots, d_n$.

$$\delta \leq d_1, \delta \leq d_2, \dots, \delta \leq d_n \text{ — (1)}$$
$$\Delta \geq d_1, \Delta \geq d_2, \dots, \Delta \geq d_n \text{ — (2)}$$

By HSP, $\sum_{i=1}^{n} d(v_i) = 2m$ — (3)

~~$d_1 + d_2 + \dots + d_n = 2$~~

$$\delta + \delta + \dots + \delta \leq d_1 + d_2 + \dots + d_n$$
$$n\delta \leq 2m$$
$$\delta \leq \frac{2m}{n}$$

$$\Delta + \Delta + \cdots + \Delta \geq d_1 + d_2 + \cdots + d_n$$

$$n\Delta \geq 2m$$

$$\Delta \geq \frac{2m}{n}$$

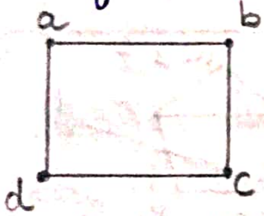
$$\therefore \delta \leq \frac{2m}{n} \leq \Delta$$

23/10/24

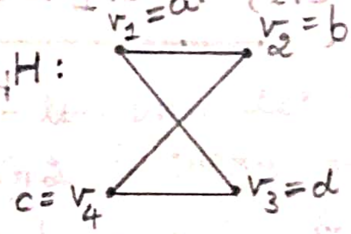
☆ Isomorphism :- Consider two graphs $G(V, E)$ & $G' = (V', E')$. Suppose there excist a function $f: V \to V'$ such that f is one to one correspondence & for all vertices a, b of G, ab is an edge of G if & only if $f(a)f(b)$ is an edge of G'. Then f is called an isomorphism between G & G' written as $G \simeq G'$.

※ I Check for isomorphism.

1. G :



H :

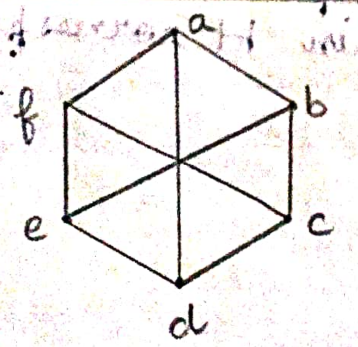


ans Graph G has 4 vertices, 4 edges & degree sequence 2, 2, 2, 2
Graph H has 4 vertices, 4 edges & degree sequence 2, 2, 2, 2

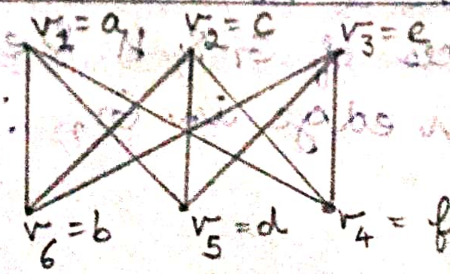
Vertex correspondence:	Edge correspondence:
$a \leftrightarrow v_1$	$ab \leftrightarrow v_1v_2$
$b \leftrightarrow v_2$	$bc \leftrightarrow v_2v_4$
$c \leftrightarrow v_4$	$cd \leftrightarrow v_4v_3$
$d \leftrightarrow v_3$	$da \leftrightarrow v_3v_1$

As there is one-to-one correspondence between vertices of G & H & every edge in H corresponds to an edge in graph G, $\therefore G \cong H$.

2. G_1 :



G_2 :



ans Graph G_1 has 6 vertices, 9 edges & deg sequence: 3, 3, 3, 3, 3, 3
Graph G_2 has 6 vertices, 9 edges & deg sequence: 3, 3, 3, 3, 3, 3

Vertex correspondence:

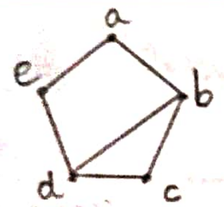
$a \leftrightarrow v_1$
$b \leftrightarrow v_6$
$c \leftrightarrow v_2$
$d \leftrightarrow v_5$
$e \leftrightarrow v_3$
$f \leftrightarrow v_4$

Edge correspondence:

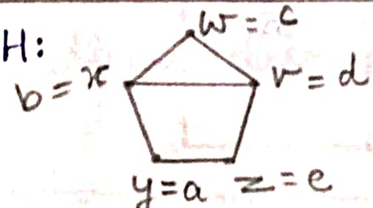
$ab \leftrightarrow v_1v_6$
$bc \leftrightarrow v_6v_2$
$cd \leftrightarrow v_2v_5$
$de \leftrightarrow v_5v_3$
$ef \leftrightarrow v_3v_4$
$fa \leftrightarrow v_4v_1$
$ad \leftrightarrow v_1v_5$
$be \leftrightarrow v_6v_3$
$cf \leftrightarrow v_2v_4$

As there is one-to-one correspondence between vertices of G_1 & G_2 & every edge in G_2 corresponds to an edge in G_1, $\therefore G_1 \simeq G_2$.

3. G:



H:



ans: G has 5 vertices, 6 edges & deg sequence 2, 2, 2, 3, 3
H has 5 vertices, 6 edges & deg sequence 2, 2, 2, 3, 3

Vertex correspondence:

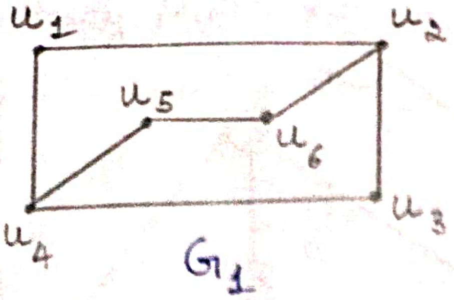
$a \leftrightarrow y$
$b \leftrightarrow x$
$c \leftrightarrow w$
$d \leftrightarrow v$
$e \leftrightarrow z$

Edge correspondence:

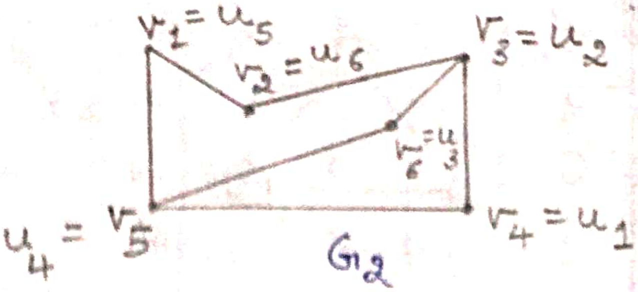
$ab \leftrightarrow yx$
$bc \leftrightarrow xw$
$cd \leftrightarrow wv$
$de \leftrightarrow vz$
$bd \leftrightarrow xv$
$ea \leftrightarrow zy$

As there is one-to-one correspondence between vertices of G & H & every edge in H corresponds to an edge in G, $\therefore G \simeq H$.

4.



G_1



G_2

ans G_1 has 6 vertices, 7 edges, deg seq.: 2, 2, 2, 2, 3, 3
G_2 has 6 vertices, 7 edges, deg seq: 2, 2, 2, 2, 3, 3

Vertex correspondence:

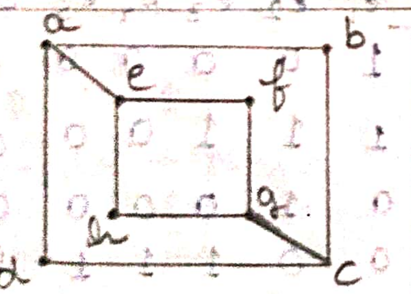
$u_1 \Leftrightarrow v_4$
$u_2 \Leftrightarrow v_3$
$u_3 \Leftrightarrow v_6$
$u_4 \Leftrightarrow v_5$
$u_5 \Leftrightarrow v_1$
$u_6 \Leftrightarrow v_2$

Edge correspondence:

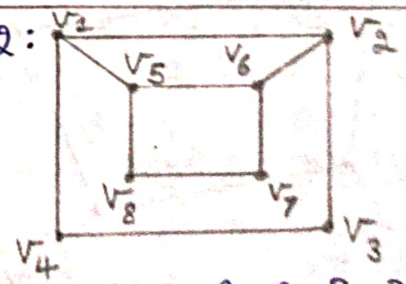
$u_1u_2 \Leftrightarrow v_4v_3$
$u_2u_3 \Leftrightarrow v_3v_6$
$u_3u_4 \Leftrightarrow v_6v_5$
$u_4u_1 \Leftrightarrow v_5v_4$
$u_2u_6 \Leftrightarrow v_3v_2$
$u_6u_5 \Leftrightarrow v_2v_1$
$u_5u_4 \Leftrightarrow v_1v_5$

As there is one-to-one correspondence between vertices of G1 & G2 & every edge in G2 corresponds to an edge in G1, $\therefore G1 \simeq G2$.

5. G1:



G2:



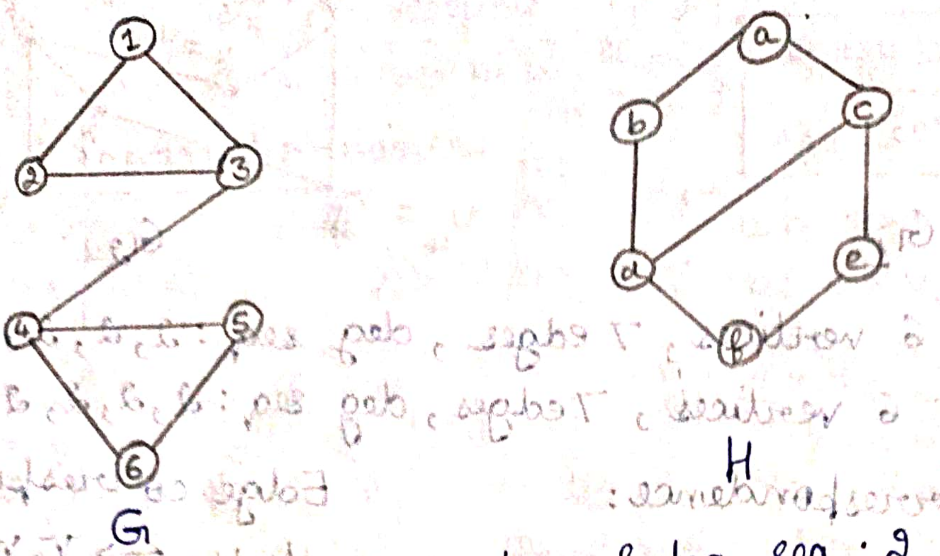
G1 has 8 vertices, 10 edges & deg. seq.: 2, 2, 2, 2, 3, 3, 3, 3
G2 has 8 vertices, 10 edges & deg. seq.: 2, 2, 2, 2, 3, 3, 3, 3

a is degree 3 vertex adjacent to 2 vertices of deg. 2 & 1 vertex of deg. 3 in G1.

In G2, we cannot find a deg. 3 vertex adjacent to 2 vertices of deg. 2 & 1 vertex of deg. 3.

So, there is no one-to-one correspondence between vertices of G1 & G2. $\therefore G_1 \not\simeq G_2$.

6.

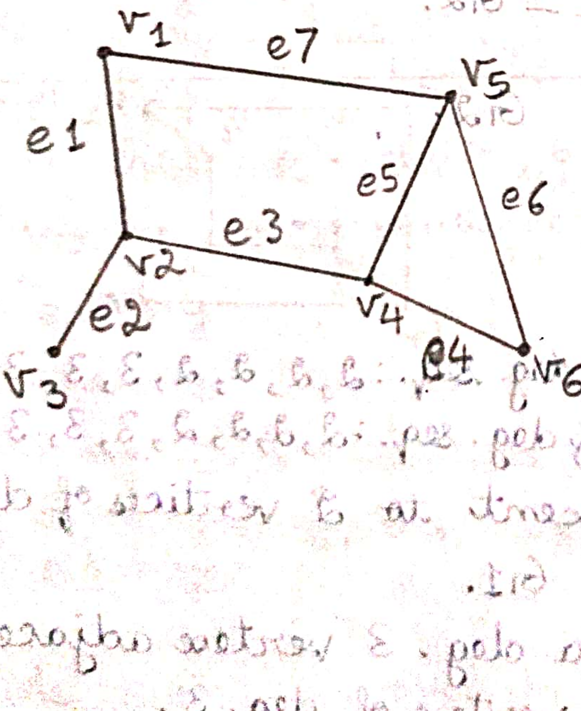


G has 6 vertices, 7 edges & deg. seq.: 2, 2, 2, 2, 3, 3
H has 6 vertices, 7 edges & deg. seq.: 2, 2, 2, 2, 3, 3
In G, ③ is a degree 3 vertex adjacent to three degree 2 vertices. But we cannot find such a vertex in H.
∴ There is no one-to-one correspondence between vertices of G & H. ∴ $G \not\cong H$.

☆ Matrix representation of a graph :-

1. Incidence Matrix:

Eg:

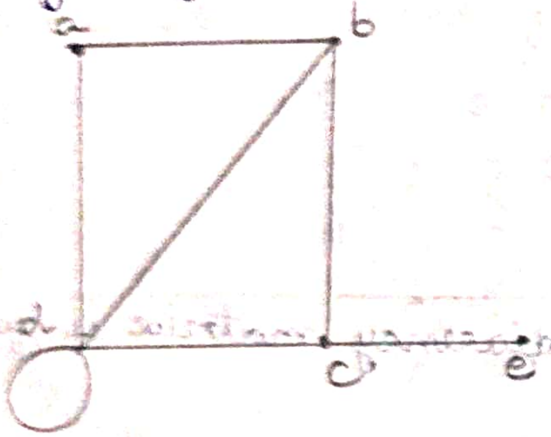


Incidence matrix:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}_{6\times 7} \end{array}$$

2. Adjacency Matrix:

Eg:



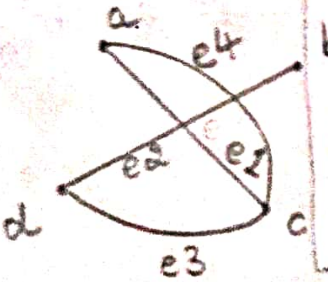
$$X = \begin{array}{c|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{5\times5} \end{array}$$

☆ Problems:-

1 Draw graph G whose incidence matrix is given by

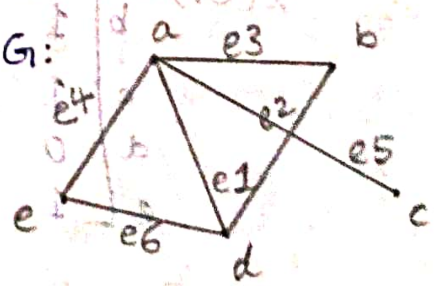
i)
$$\begin{array}{c|c} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

ans G:



ii)
$$\begin{array}{c|c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

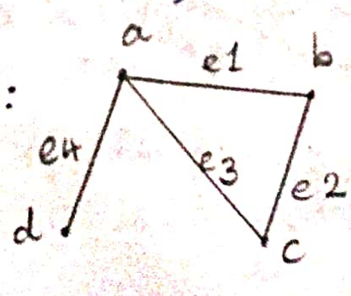
ans: G:



2. For the given incidence matrix, obtain the adjacency matrix.

i)
$$\begin{array}{c|c} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

G:



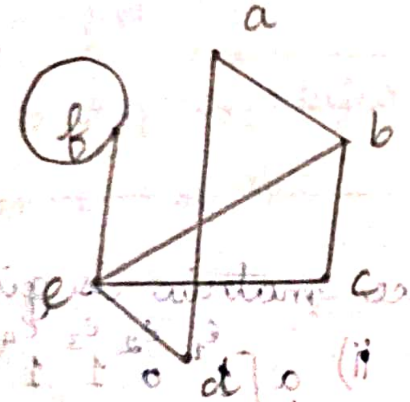
Adjacency matrix is:

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3. Construct a graph whose adjacency matrix is given by

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

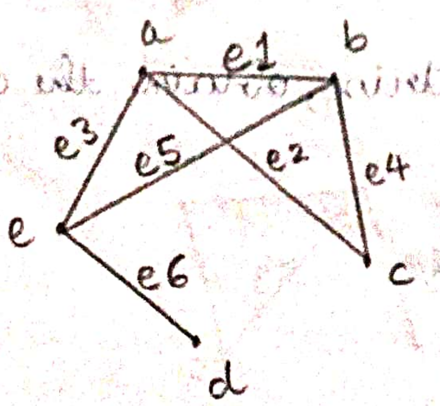
ans G:



4. For the given adjacency matrix, construct incidence matrix.

$$x(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

ans G:



Incidence matrix is:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{cccccc} e1 & e2 & e3 & e4 & e5 & e6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

5. Without actually constructing the graph, show that there exist no connected graph whose incidence matrix is :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

ans $C_3 \Leftrightarrow C_6 \Rightarrow \begin{bmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{bmatrix} \Rightarrow \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$

$\because$ The resultant matrix is of the form $\begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$,

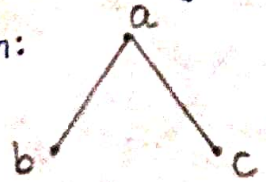
where $A(G_1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ & $A(G_2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the graph for the given matrix is a disconnected graph.

★ **Subgraph**: Given two graphs G_1 & G we say G_1 is a subgraph of G denoted by $G_1 \subseteq G$ if following conditions hold:

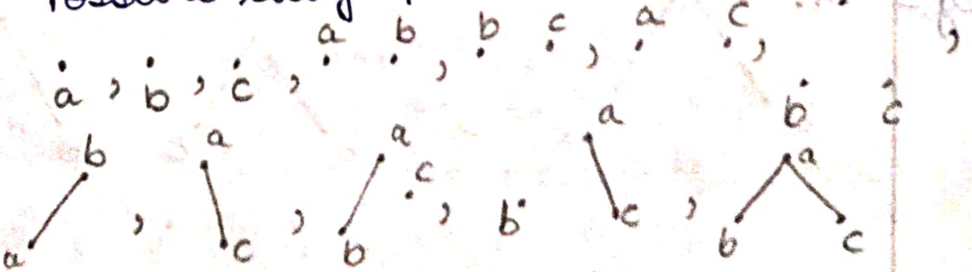
(i) All the vertices & all the edges of G_1 are in G

(ii) Each edge of G_1 has same end vertices in G as in G_1.

Eg:- G:



Possible subgraphs:



NOTE:- i) A subgraph of a graph is also a graph.

ii) Any graph isomorphic to a subgraph of graph G is also a subgraph of G.

iii) Every graph is a subgraph of itself.

iv) A single vertex in a graph G is a subgraph of G.

v) A single edge in a graph G together with its end vertices is a subgraph of G.

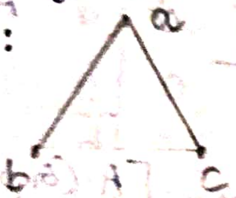
vi) If $G_1 \subseteq G_2$ & $G_2 \subseteq G_3$, then $G_1 \subseteq G_3$.

☆ Types of Subgraph:-

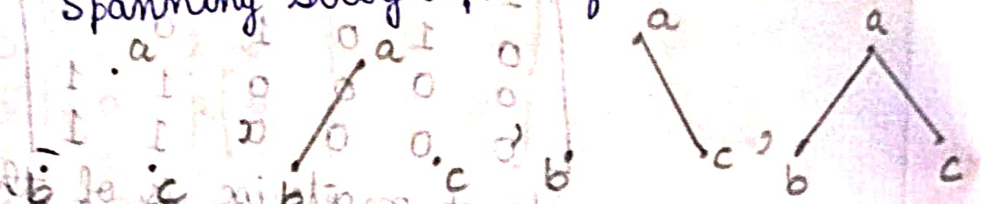
1. Spanning subgraph: Given a graph $G(V, E)$ if there is a subgraph $G_1(V_1, E_1)$ of G such that $V_1 = V$, then G_1 is a spanning subgraph of G.

NOTE: Every graph is its own spanning subgraph.

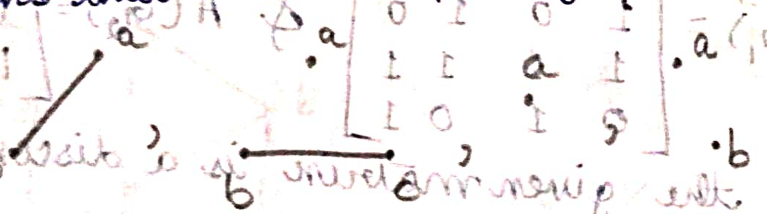
Eg:- G:



Spanning subgraphs of G:

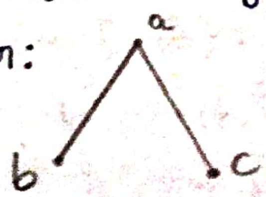


Subgraphs that are not Spanning:

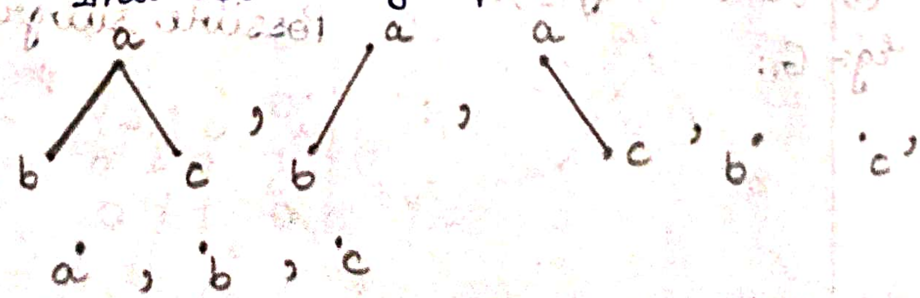


2. Induced subgraph: Given a graph $G(V, E)$ if there is a subgraph $G_1(V_1, E_1)$ of G such that every edge ab of G, where $a, b \in V_1$ is an edge of G_1 also. Then G_1 is called an induced subgraph of G induced by V_1 denoted by $\langle V_1 \rangle$.

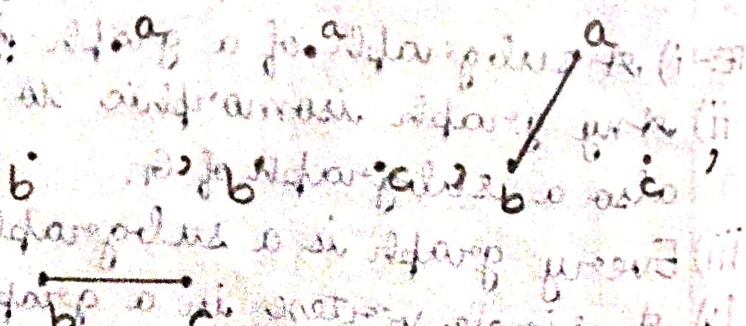
Eg:- G:



Induced subgraph:



Not induced subgraphs:

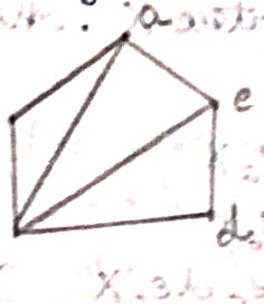


★ **Edge disjoint & vertex disjoint:** Let G be a graph & G_1 & G_2 be two subgraphs of G. Then

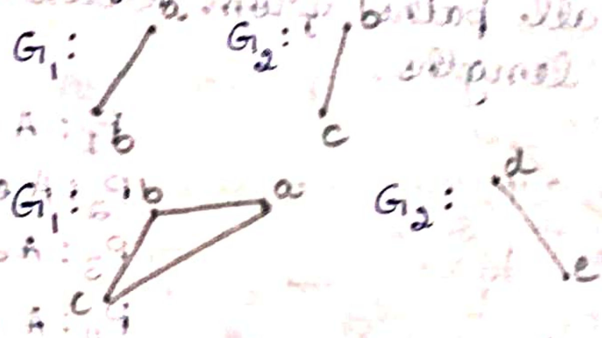
(i) G_1 & G_2 are said to be edge disjoint if they do not have any edge in common.

(ii) G_1 & G_2 are said to be vertex disjoint if they do not have any edge & vertex in common.

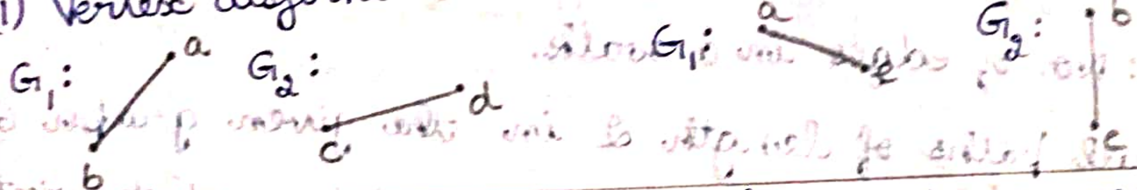
Eg:- G:



(i) Edge disjoint:

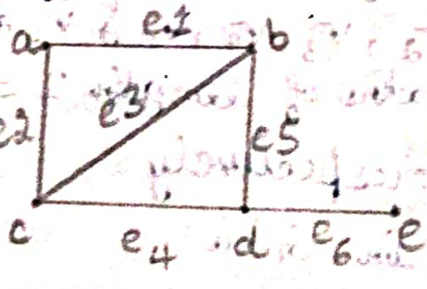


(ii) Vertex disjoint:



★ **Walk & its classifications:** Let G be a graph having at least one edge. In G, consider a finite alternating sequence of vertices & edges of the form $v_i e_j v_{i+1} e_{j+1} v_{i+2} \ldots e_k v_m$ which begins & ends with vertices & each edge in the sequence is incident on the vertices preceding & following it in sequence. Such a sequence is called a walk in G.

Eg:- G:



w_1: $a e_1 b e_5 d$ → open, Trail, path

w_2: $a e_1 b$ → O, T, P

w_3: $a e_1 b e_5 d e_6 e$ → O, T, P

w_4: $a e_1 b e_3 c e_2 a$ → Closed, Circuit, cycle

w_5: $a e_1 b e_3 c$ → O, T, P

w_6: $a e_1 b e_1 a$ → Closed

w_7: $a e_1 b e_3 c e_2 a e_1 b e_5 d$ → O

→ A walk that begins & ends at different vertex is called an open walk, otherwise a closed walk.

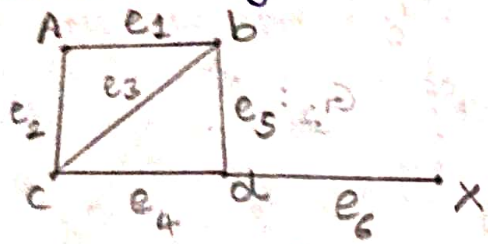
→ An open walk in which no edges repeat is called a trail.

→ A closed walk in which no edges repeat is called a circuit.

→ A trail in which no vertex repeats is called a path

→ A circuit in which no vertex repeats (other than initial vertex where initial vertex do not appear as internal vertex) is called a cycle.

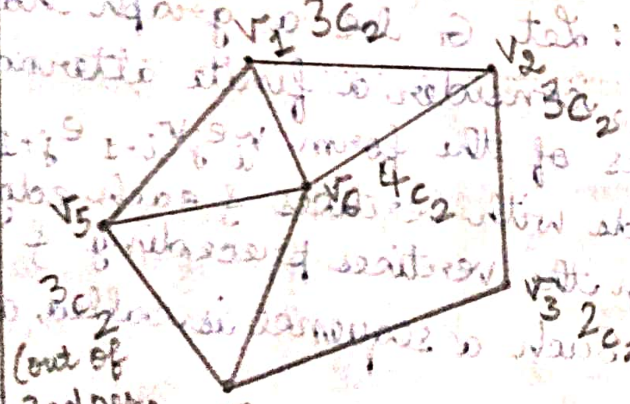
1Q. Find all paths from vertex A to vertex X. Also specify their lengths.



	Length
$P_1 : A e_1 b e_5 d e_6 X$	3
$P_2 : A e_2 C e_4 d e_6 X$	3
$P_3 : A e_2 C e_3 b e_5 d e_6 X$	4
$P_4 : A e_1 b e_3 C e_4 d e_6 X$	4

NOTE:- Length : No. of edges in a walk.

2. Find all paths of length 2 in the given graph G.



(out of 3 edges, choose 2, as length = 2) $3c_2$

(Fix the intermediate vertex)

∵ We are looking for paths of length 2, we shall fix the middle vertex.

If v_1 is the middle vertex, then there are $3c_2$ ways of getting paths of length 2.

Similarly if v_2, v_3, v_4, v_5 & v_6 is the middle vertex, then no. of paths of length 2 = $3c_2, 2c_2, 3c_2, 3c_2, 4c_2$ respectively

∴ No. of paths of length 2 in G is

$$= 3c_2 + 3c_2 + 2c_2 + 3c_2 + 3c_2 + 4c_2$$

$$= 3+3+1+3+3+6$$

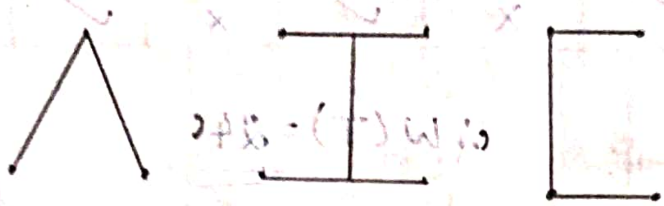
$$= \underline{\underline{19}}$$

★ Connected graph : If there is a path between two vertices a & b, then a & b are said to be connected.

A graph in which every two vertices are connected, is called connected graph.

★ Tree : A graph G is said to be tree if it is connected & has no cycles. Tree is denoted by T.

Eg:-

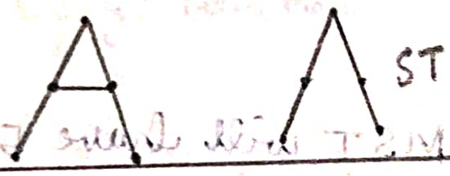


NOTE: i) In tree there is one & only one path between every pair of vertices

ii) A tree with n vertices will have $n-1$ edges.

★ Spanning Tree : Let G be a connected graph. A subgraph T of G is called spanning tree of G if (i) T is a tree & (ii) T contains all vertices of G.

Eg:- G:



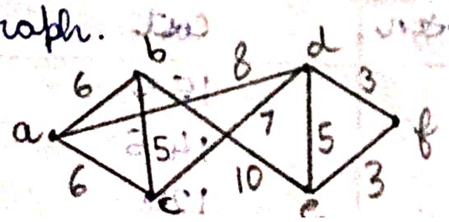
★ Minimal Spanning tree : Let G be a graph & suppose there is a positive real number associated with each edge of G, then G is called weighted graph & the positive real number is called a weight.

Let G be a weighted graph & T be its spanning tree. Every branch of T is an edge of G which has some weight. The sum of weight of all the branches of T is called weight of T denoted by W(T).

Suppose we consider all spanning trees of a connected weighted graph & find weights of every spanning tree. A spanning tree with least weight is minimal spanning tree. This tree is not unique.

I Use Kruskal's algorithm to find minimal spanning tree of weighted graph.

1.

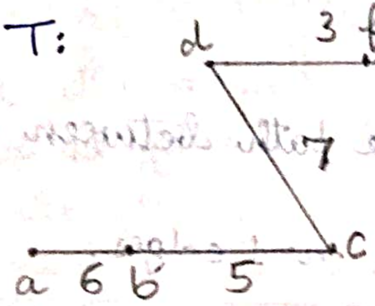


1S.

ans The graph has 6 vertices, so MST will have 5 edges

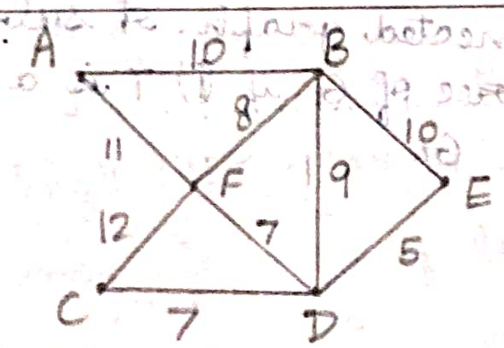
Edge	df	ef	bc	de	ab	ac	cd	ad	6o
Weight	3	3	5	5	6	6	7	8	10
Select	✓	✓	✓	X	✓	X	✓	—	

T:



$\therefore W(T) = \underline{\underline{24}}$

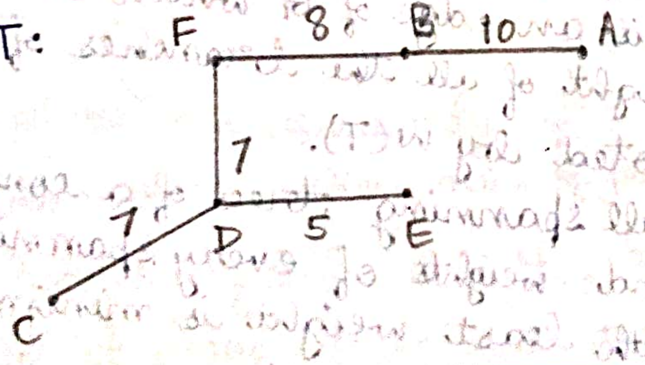
2.



ans The graph has 6 vertices, so MST will have 5 edges

Edge	DE	CD	FD	FB	BD	BE	AB	AF	CF ~~BE~~
Weight	5	7	7	8	9	10	10	11	12
Select	✓	✓	✓	✓	X	X	✓	X	

T:



$W(T) = 37$

3. Eight cities A, B, C, D, E, F, G, H are required to be connected by new railway network. Possible tracks & the cost involved to lay them are given below:

Track between	Cost
A & B	155
A & D	145
A & G	120

B & C	145
C & D	150
C & E	95
D & F	100
E & F	150
F & G	140
F & H	150
G & H	160

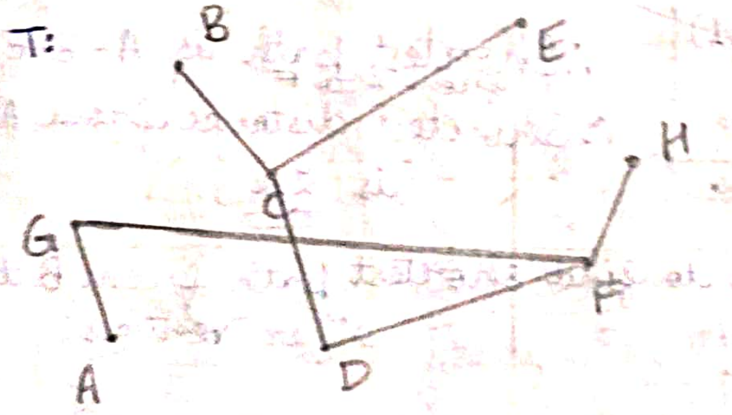
Construct a railway network connecting 8 cities involving minimum cost.

ans

Edge	CE	DF	AG	FG	AD	BC	CD	EF	FH	AB	GH
Weight	95	100	120	140	145	145	150	150	150	155	160
Select	✓	✓	✓	✓	✗	✓	✓	✗	✓	✗	✗

Railway tracks are 7.

T:



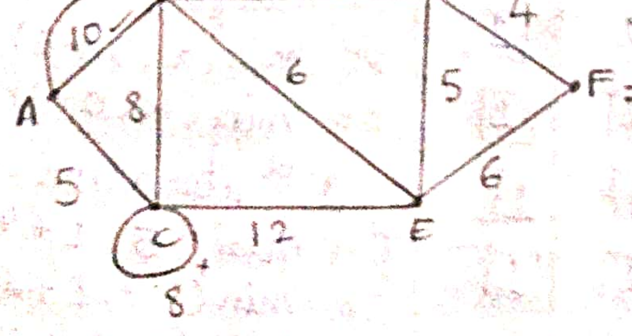
$w(T) = 900$

∴ Minimum cost is ₹900 cr.

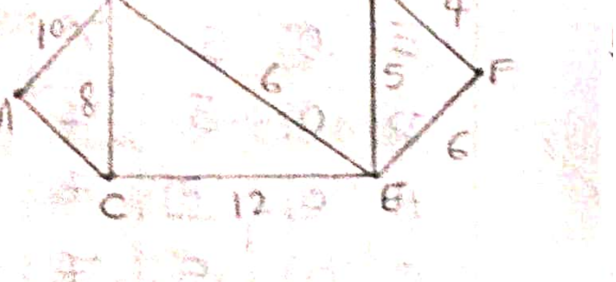
★ Dijkstra's Algorithm :-

min (temp label; (permanent label + edge wt))

1. Find shortest path from A to F using Dijkstra's algorithm



⇒



ans

	A	B	C	D	E	F
A	$\boxed{0}$	∞	∞	∞	∞	∞
C	$\boxed{0}$	10	$\boxed{5}$	∞	∞	∞
B	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	∞	17	∞
E	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	22	$\boxed{16}$	∞
D	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	$\boxed{21}$	$\boxed{16}$	22
F	$\boxed{0}$	$\boxed{10}$	$\boxed{5}$	$\boxed{21}$	$\boxed{16}$	$\boxed{22}$

1st iteration:- (PL=0)

AB min (∞, 0+10)

AC min (∞, 0+5)

AD min (∞, 0+∞)

AE min (∞, 0+∞)

AF min (∞, 0+∞)

2nd iteration (PL=5)

BC min (10, 5+8)

CD min (∞, 5+∞)

CE min (∞, 5+12)

CF min (∞, 5+∞)

3rd iteration (PL=10)

BD min (∞, 10+12)

BE min (17, 10+6)

BF min (∞, 10+∞)

4th iteration (PL=16)

ED min (22, 16+5)

EF min (∞, 16+6)

(—‖— from A to D : A-B-E-D
—‖— distance : 21)

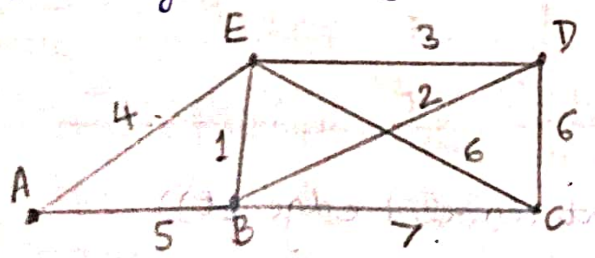
5th iteration (PL=21)

DF min (22, 21+4)

∴ Shortest path from A to F is A-B-E-F

∴ Shortest distance from A to F is <u>22</u>

2\. Use Dijkstra's algorithm to find shortest path from B to all other vertices.



ans

	B	A	C	D	E
B	$\boxed{0}$	∞	∞	∞	∞
E	$\boxed{0}$	5	7	2	$\boxed{1}$
D	$\boxed{0}$	5	7	$\boxed{2}$	$\boxed{1}$
A	$\boxed{0}$	$\boxed{5}$	7	$\boxed{2}$	$\boxed{1}$
C	$\boxed{0}$	$\boxed{5}$	7	$\boxed{2}$	$\boxed{1}$

1st iteration:- (PL=0)

BA min (∞, 0+5)

BC min (∞, 0+7)

BD min (∞, 0+2)

BE min (∞, 0+1)

2nd iteration :- (PL=1)

EA min (5, 1+4)

EC min (7, 1+6)

ED min (2, 1+3)

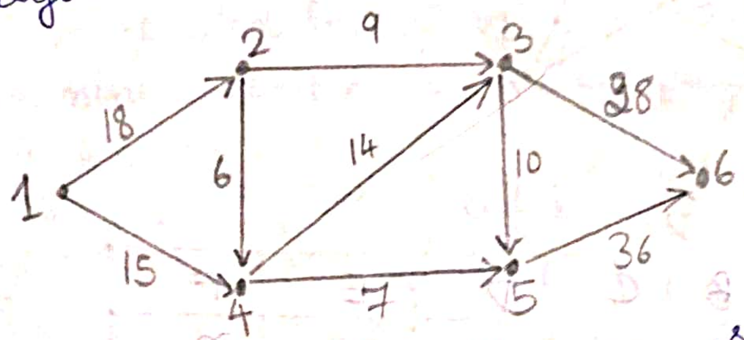
3rd iteration:- (PL= 2)
DA min (5, 2 + ∞)
DC min (7, 2+ 6)

4th iteration:- (PL=5)
AC min (7, 5+∞)

	Shortest path	Shortest distance
B : E	B – E	1
B : D	B – D	2
B : C	B – C	7
B : A	B – A	5

3. Find shortest path from 1 to all remaining vertices using Dijkstra's algorithm.

Table -5m-



	1	2	3	4	5	6
1	$\boxed{0}$	∞	∞	∞	∞	∞
4	$\boxed{0}$	18	∞	$\boxed{15}$	∞	∞
2	$\boxed{0}$	$\boxed{18}$	29	$\boxed{15}$	22	∞
5	$\boxed{0}$	$\boxed{18}$	27	$\boxed{15}$	$\boxed{22}$	∞
3	$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\boxed{15}$	$\boxed{22}$	58
6	$\boxed{0}$	$\boxed{18}$	$\boxed{27}$	$\boxed{15}$	$\boxed{22}$	$\boxed{55}$

1st iteration:- (PL=0)
1-2 : min (∞ , 0+18)
1-3 : min (∞ , 0+ ∞)
1-4 : min(∞ , 0+15)
1-5 : min (∞, 0+ ∞)
1-6 : min (∞, 0+∞)

2nd iteration:- (PL=15)
4-2 : min (18, 15+∞)
4-3 : min (∞, 15+ 14)
4-5 : min (∞, 15+7)
4-6 : min (∞, 15+∞)

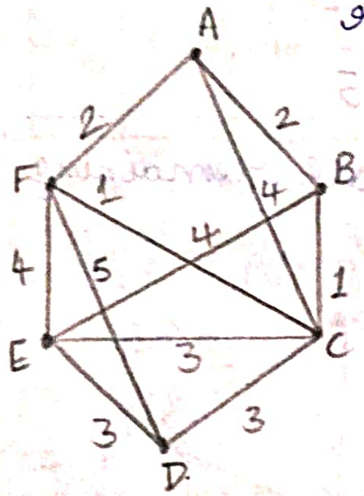
3rd iteration :- (PL=18)
2-3 : min (29, 18+9)
2-5 : min (22, 18+∞)
2-6 : min (∞ , 18 +∞)

4th iteration :- (PL= 22)
5-3 : min (27, 22+ ∞)
5-6 : min (∞, 22+36)

5th iteration:- (PL= 27)
3-6 : min (58, 27+28)

	Shortest path	Shortest distance
1-2	1-2	18
1-3	1-2-3	27
1-4	1-4	15
1-5	1-4-5	22
1-6	1-2-3-6	55

4. Find shortest path & shortest distance from A to all remaining vertices using Dijkstra's algorithm.



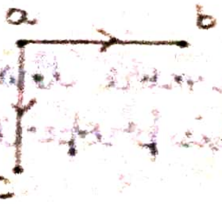
	A	B	C	D	E	F
A	$\boxed{0}$	∞	∞	∞	∞	∞
B	$\boxed{0}$	$\boxed{2}$	4	∞	∞	2
F	$\boxed{0}$	$\boxed{2}$	3	∞	6	$\boxed{2}$
C	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	7	6	$\boxed{2}$
D	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{6}$	6	$\boxed{2}$
E	$\boxed{0}$	$\boxed{2}$	$\boxed{3}$	$\boxed{6}$	$\boxed{6}$	$\boxed{2}$

	Shortest path	Shortest distance
A:F	A-F	2
A:E	A-B-E	6
A:D	A-B-C-D	6
A:C	A-B-C	3
A:B	A-B	2

★ Directed graph (Digraph):-

A directed graph or digraph is a pair (V, E), where V is a nonempty set & E is a set of ordered pairs of elements taken from the set V.

Eg:- D:



$V = \{a, b, c\}$

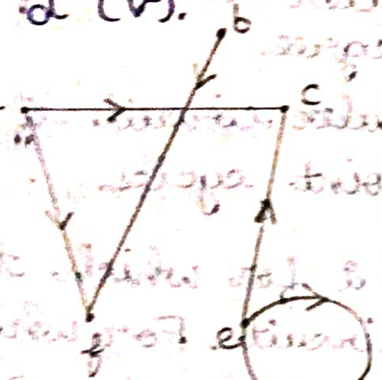
$E = \{ab, ac\}$

NOTE:- vertex from where edge starts is called initial vertex & where it ends is called terminal vertex.

★ Indegree & outdegree:-

If v is a vertex of a digraph D, the no. of edges for which v is the initial vertex is called outdegree of v denoted by $d^+(v)$ & no. of edges for which v is the terminal vertex is called indegree of v denoted by $d^-(v)$.

Eg- D:



*	a	b	c	d	e	f
5 indeg	0	0	2	0	1	2
,5 outdeg	2	1	0	0	2	0

= m(5) no. of edges

Indegree + Outdegree = Degree of a vertex

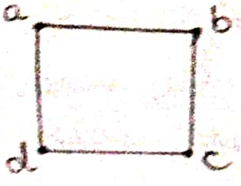
NOTE:- 1) Indegree of v + Outdegree of v = Degree of v

2) Sum of all outdegrees of all vertices is equal to the sum of indegrees of all vertices, each sum being equal to number of edges in D.

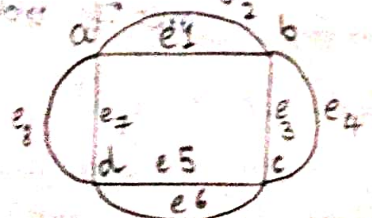
★ Eulerian graph or Euler graph:-

Consider a connected graph G. If there is a circuit in G that contains all edges of G, then that circuit is called Eulerian circuit in G. If there is a trail in G that contains all edges of G, then that trail is called a Eulerian trail in G.

Eg:-



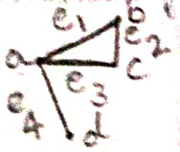
E.C. abcda



E.C. $a e_1 b e_3 c e_5 d e_7 a e_2 b e_4 c e_6 d e_8 a$

NOTE:- If deg. of every vertex in G is even, then G has Euler circuit.

Eg:-



E.T. $de_4ae_3ce_2be_1a$

NOTE:- i) In trail & circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trail & Euler circuit.

ii) Since Euler circuit & Euler trail include all edges, they will include all vertices.

iii) A connected graph that contains Euler circuit is called a Euler graph & that contains a Euler trail is called Semi-Euler graph.

iv) A connected graph G has a Euler circuit if & only if all vertices of G are of even degree.

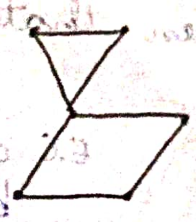
v) A connected graph G has a Euler circuit iff it can be decomposed into edge disjoint cycles.

1. Find all positive integer $n \geq 2$ for which the complete graph K_n contains a Euler circuit. For what value of n does K_n have a Euler trail but not a Euler circuit?

ans K_n for $n \geq 2$ has Euler circuit for n = odd (deg. of every vertex is $n-1$ which is equal to even).
K_n has Euler trail for $n=2$.

2. (a) Does an Euler graph exist with even number of vertices & odd no. of edges?
(b) Does Euler graph exist with odd number of vertices & even no. of edges?

ans (a) Yes



$n = 6$ even
$m = 7$ odd

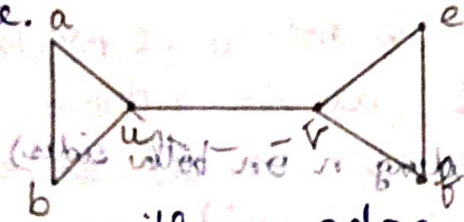
(b) Yes.

$n = 5$
$m = 6$

3. Show that a connected graph with exactly two vertices of odd degree has an Euler trail.

ans Let G be a connected graph where u & v are the only vertices with odd degree.

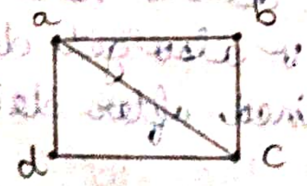
G':



Join u & v with an edge even if they are adjacent.
The resultant graph is a Euler graph as we have a Euler circuit uabuvfevu.
The new graph G' has a Euler circuit as degree of every vertex in G' is of even degree.
Deletion of this edge results in a Euler trail uabuvfev.

☆ Hamiltonian Cycle & Hamiltonian Path:-
Let G be a connected graph. If there is a cycle in G that contains all vertices of G, then that cycle is called Hamiltonian cycle in G.

Eg:- G:



HP: abcd
HC: abcda

NOTE: i) A Hamiltonian cycle in graph G of n vertices consist of exactly n edges.
ii) Hamiltonian cycle includes all vertices of G does not imply it includes all edges of G.
iii) Ore's Theorem: If in a simple connected graph with n vertices ($n \geq 3$) the sum of degrees of every pair of non-adjacent vertices is greater than or equal to n, then the graph is Hamiltonian. [Hamiltonian → Graph which has Hamiltonian cycle].

*iv) Dirac's theorem: If in a simple connected graph with n vertices ($n \geq 3$) the degree of every vertex is greater than or equal to $n/2$, then that graph is Hamiltonian graph.

Problems :-

1. Prove that a complete graph K_n where $n \geq 3$ is a Hamiltonian graph.

all vertices are adjacent, so use Dirac's theorem

ans Degree of every vertex in K_n is $n-1$. — (1)

Given $n \geq 3$

$$n > 2$$

$$2n > 2+n \quad \text{(Adding } n \text{ on both sides)}$$

$$2n-2 > n$$

$$n-1 > n/2$$

$$\deg(v) > n/2$$

$\therefore$ By Dirac's theorem, K_n is Hamiltonian.

*2. Let G be a simple graph with n vertices & m edges where m is at least 3. If $m \geq \frac{(n-1)(n-2)}{2} + 2$, prove that G is Hamiltonian graph. Is the converse true?

ans Let G have n vertices & m edges ($m \geq 3$). Let u & v be two non adjacent vertices in G.

To prove that: $\deg u + \deg v \geq n$

Let $\deg u = x$ & $\deg v = y$. Suppose we delete vertices u & v from G, edges incident on u & v also get deleted. Number of edges in G' [graph obtained after deletion] is

$$q = m - x - y.$$

Pg 5

$$x + y = m - q$$

$$\left(\frac{n(n-1)}{2}\right)$$

$$\deg u + \deg v \geq \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2}$$

{G' has q edges & $n-2$ vertices, so $q \leq \frac{(n-2)(n-3)}{2}$}

$$\geq \frac{n-2}{2}[n-1-n+3] + 2$$

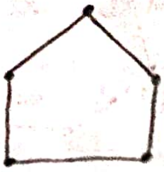
$$\geq \frac{n-2}{\cancel{2}}[\cancel{2}] + 2$$

$$\deg u + \deg v \geq n$$

$\therefore$ By Ore's theorem, G is Hamiltonian graph.

Converse

G:



$n=5$
$m=5$

G is Hamiltonian with $m=5$

$m \geqslant \frac{(4)(3)}{2} + 2$

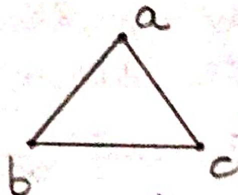
$m \geqslant 8$

$\therefore$ Converse is false

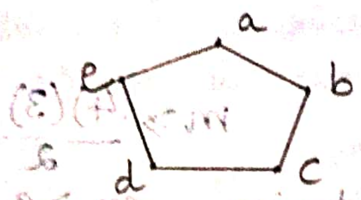
3. Suppose a new club has an odd number of members say $n = 2k+1$, where k is a positive integer. These members meet each day for lunch at round table. They decide to sit in such a way that every member has different neighbors at each lunch. How many days can this arrangement last?

ans Let us consider a graph G in which a member x is represented by a vertex & possibility of his sitting next to another member y is represented by an edge between x & y. Since every member is allowed to sit next to any other member, G is a complete graph. Since there are n members, these are n vertices. Every sitting arrangement around a table is Hamiltonian cycle. On Day 1, they can sit in any order, this will form Hamiltonian cycle C1. On second day, If they are to sit such that every member has different neighbour, we must find Hamiltonian cycle C2 which is edge disjoint with C1. If the same arrangement has to be there on subsequent days, then each day we have to find Hamiltonian cycle which is edge disjoint with Hamiltonian cycles found earlier. No. of such cycles is $\frac{n-1}{2}$. Given $n = 2k+1$, no. of such arrangements will be k.

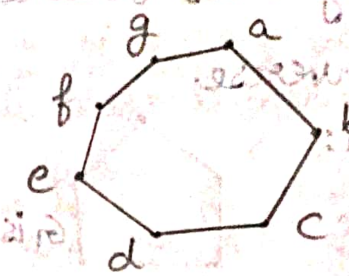
Eg:- $n=3$ $n=5$ $n=7$



Day 1
abca



Day -1
abcdea



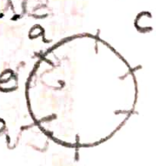
Day-1: abcdefga
Day-2: acegbdfa
Day-3: adgcfbea

Day-2
acebda

☆ Problems on connectedness:-

1. Prove that a connected graph G remains connected after removing an edge e from G iff e is a part of some cycle in G.

ans. Suppose e is part of some cycle C of G. Then the end vertices of e say 'a' & 'b' are connected by atleast two paths one of which is e & other is C-e.



Hence removal of e from G will not affect connectivity of G because even after removal of e, the end vertices are connected by C-e. conversely suppose e is not a part of some cycle in G. Then end vertices of e are connected by atmost one path. Hence, removal of e from G disconnects these end points. This implies G-e is a disconnected graph.

If ~~G-e is~~ e is not a part of cycle in G, then G-e is disconnected.

$\Rightarrow$ if G-e is connected, then e has to be part of cycle in G.

2. If G is a simple graph in which degree of every vertex is atleast $\frac{n-1}{2}$, prove that G is connected.

ans Take any two vertices u & v of G. They are either adjacent or non-adjacent: If u & v are adjacent, then G is connected. Suppose u & v are non-adjacent then each will have atleast $\frac{n-1}{2}$ neighbours.

$\therefore$ u & v together will have atleast $n-1$ neighbours. But since G has n vertices & u & v are not adjacent, ~~total~~ no. of neighbours u & v have ~~together~~ is only $n-2$. Therefore atleast one vertex, say 'x', is neighbour of u & v. Hence there is an edge between u & x, x & v i.e. there is a path between u & v.
So, G is connected.

3. Let G be a disconnected graph of even order n with two components each of which is complete. Prove that G has minimum of $\frac{n(n-2)}{4}$ edges.

ans Let there be two components of G. Let x be no. of vertices in G_1 & $n-x$ be no. of vertices in G_2.
Given G_1 & G_2 are complete graph.

No. of edges in $G_1 = \frac{x(x-1)}{2}$

No. of edges in $G_2 = \frac{(n-x)(n-x-1)}{2}$

No. of edges in $G = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$

$$f(x) = \frac{n^2 - 2nx + 2x^2 - n}{2}$$

$$f'(x) = \frac{-2n + 4x}{2} = 0$$

$$x = n/2$$

$$f''(x) = 2$$

$$f''\left(\frac{n}{2}\right) = 2 > 0$$

[$f(x)$ attains minimum at $x = n/2$]

Min. value of $f(x)$ is $\frac{n^2 - 2n\cdot\frac{n}{2} + 2\cdot\frac{n^2}{4} - n}{2}$

$$= \frac{\frac{n^2}{2} - n}{2} = \frac{n^2 - 2n}{4} = \frac{n(n-2)}{4}$$