**B.M.S. COLLEGE OF ENGINEERING, BENGALURU – 560 0 19**
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**DEPARTMENT OF MATHEMATICS**

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# STATISTICS AND DISCRETE MATHEMATICS

## (Course Code: 23MA3BSSDM)

## UNIT-3: JOINT PROBABILITY AND MARKOV CHAINS

### Joint Probability: Discrete Case

1. The joint probability distribution of two random variables $X$ and $Y$ is given below:

| Y<br>X | 2 | 3 | 4 |
|---|---|---|---|
| 1 | 0.06 | 0.15 | 0.09 |
| 2 | 0.14 | 0.35 | 0.21 |

Find the marginal distributions of $X$ and $Y$. Also verify whether $X$ and $Y$ are stochastically independent.

2. The joint probability distribution of two random variables $X$ and $Y$ is given below:

| Y<br>X | -3 | 2 | 4 |
|---|---|---|---|
| 1 | 0.1 | 0.2 | 0.2 |
| 3 | 0.3 | 0.1 | 0.1 |

Find:
(i) the marginal distributions of $X$ and $Y$
(ii) $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and $Cov\ (X, Y)$

3. The joint probability distribution of two random variables $X$ and $Y$ is given below:

| Y<br>X | -4 | 2 | 7 |
|---|---|---|---|
| 1 | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{8}$ |
| 5 | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |

Find: (i) the marginal distributions of $X$ and $Y$ (ii)$E(X)$ and $E(X)$ (iii) $\sigma_X, \sigma_Y$ (iv)$Cov\ (X, Y)$ (v) $\rho(X, Y)$

4. The joint probability distribution of two random variables $X$ and $Y$ is given below:

| Y<br>X | 1 | 3 | 9 |
|---|---|---|---|
| 2 | $\frac{1}{8}$ | $\frac{1}{24}$ | $\frac{1}{12}$ |
| 4 | $\frac{1}{4}$ | $\frac{1}{4}$ | 0 |
| 6 | $\frac{1}{8}$ | $\frac{1}{24}$ | $\frac{1}{12}$ |

Find the marginal distributions of $X$ and $Y$. Also evaluate $Cov\ (X, Y)$.

5. The joint probability function of two random variables $X$ and $Y$ is given by $f(x, y) = c(2x + y)$ where $x$ and $y$ can assume all integral values such that $0 \leq x \leq 2, 0 \leq y \leq 3$ and $f(x, y) = 0$, otherwise. Find:

(i) the value of the constant $c$
(ii) $P(X \geq 1, Y < 2)$
(iii) Marginal Probability distribution of $X$ and $Y$.
(iv) Check whether $X$ and $Y$ are independent.

6. A coin is tossed three times. Let $X$ denote 0 or 1 according as tail or head occurs on the first toss. Let $Y$ denote the total number of tails which occur. Determine:
   (i) the marginal distributions of $X$ and $Y$
   (ii) the joint distributions of $X$ and $Y$. Also find the expected values of $X + Y$ and $XY$.

7. Two flashcards are selected at random from a box which contains five cards numbered $1, 1, 2, 2$ and $3$. Find the joint distributions of X and Y where X denotes the sum of two numbers and Y denote the maximum of two numbers drawn. Also determine $Cov\,(X, Y)$.

8. $X$ and $Y$ are independent random variables. $X$ takes values $2, 5, 7$ with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. $Y$ takes values $3, 4, 5$ with probabilities $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$.
   (i) Find the joint probability distribution of $X$ and $Y$
   (ii) Show that the covariance of $X$ and $Y$ is equal to zero
   (iii) Find the probability distribution of $Z = X + Y$

9. Two fruits are selected at random from a bag containing 3 Apples, 2 Oranges and 4 Mangoes. If X and Y are respectively, the number of Apples and the number of Oranges included among the two fruits drawn from the bag, find the probability associated with all possible pair of values $(x, y)$. Also find the correlation between the variables $X$ and $Y$.

10. Two marbles are selected at random from a box containing 3 Blue, 2 Red and 3 Green marbles. If X and Y are respectively, the number of Blue marbles and the number of Red marbles selected, find:
    (i) the probability associated with all possible pair of values
    (ii) Show that the random variables X and Yare not stochastically independent.

## Joint Probability: Continuous Case

1. Find the constant $k$ so that
$$f(x, y) = \begin{cases} k(x+1)e^{-y}, & 0 < x < 1, y > 0 \\ 0 & elsewhere \end{cases}$$
is a joint probability density function. Are $x$ and $y$ independent?

2. Suppose the joint p.d.f of $(x, y)$ is given by
$$p(x, y) = \begin{cases} e^{-y}, & x > 0, y > x \\ 0 & elsewhere \end{cases}$$
Find: (i) the marginal density function of $x$ (ii) the marginal density function of $y$
Check whether $x$ and $y$ are independent.

3. Let $x$ and $y$ be random variables having the joint density function
$$f(x, y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$
Find the marginal density functions of $x$ and $y$. Verify that $E(x + y) = E(x) + E(y)$ and $E(xy) = E(x)E(y)$

4. The joint probability density function of two random variables $x$ and $y$ is given by
$$p(x,y)=\begin{cases}2, & 0<x<y<1\\ 0 & elsewhere\end{cases}$$
Find the marginal density functions of $x$ and $y$. Also find the covariance between $x$ and $y$.

5. The joint density function of two random variables $x$ and $y$ is given by
$$f(x,y)=\begin{cases}kxy, & 0<x<4, 1<y<5\\ 0 & elsewhere\end{cases}$$
Find: (i) the value of $k$ (ii) $E(x), E(y), E(xy), E(2x+3y)$

6. If the joint probability function for $(x,y)$ is
$$f(x,y)=\begin{cases}c(x^2+y^2), & 0\le x\le 1, 0\le y\le 1, c\ge 0\\ 0 & otherwise\end{cases}$$
Determine the following: (i) the value of the constant $c$ (ii) marginal density functions of $x$ and $y$ (iii) $P\left(x<\frac{1}{2}, y>\frac{1}{2}\right)$ (iv) $P\left(\frac{1}{4}<x<\frac{3}{4}\right)$ (v) $P\left(y<\frac{1}{2}\right)$

7. The joint probability function for $(x,y)$ is given by
$$f(x,y)=\begin{cases}cxy, & 0\le x\le 2, 0\le y\le x\\ 0 & otherwise\end{cases}$$
Determine $c$ and hence evaluate $P\left(\frac{1}{2}<x<1\right)$

8. Verify that $f(x,y)=\begin{cases}e^{-(x+y)}, & x\ge 0, y\ge 0\\ 0 & otherwise\end{cases}$
is a density function of a joint probability distribution. Hence evaluate:
(i) $P\left(\frac{1}{2}<x<2, 0<y<4\right)$ (ii) $P(x<1)$ (iii) $P(x\ \le\ y)$ (iv) $P(x>y)$
(v) $P(x+y\ \le\ 1)$

9. Suppose that the joint density function of two random variables $x$ and $y$ is
$$f(x,y)=\begin{cases}x^2+\frac{1}{3}xy, & 0<x<1, 0<y<2\\ 0 & elsewhere\end{cases}$$
Evaluate the following:
(i) $P\left(x\ \ge \frac{1}{2}\right)$ (ii) $P(x+y\ \ge\ 1)$ (iii) $P(Y\ \le\ x)$

10. For the distribution given by the density function
$$f(x,y)=\begin{cases}\frac{1}{96}xy, & 0<x<4, 1<y<5\\ 0 & otherwise\end{cases}$$
Evaluate: (i) $P(1<x<2, 2<y<3)$ (ii) $P(x\ \ge\ 3, y\ \le\ 2)$ (iii) $P(x+y\ \le\ 3)$ (iv) $P(x+y\le 3)$

1. Alice and Bob meet for lunch every day at a random time between noon and 1 P.M. Bob always arrives after Alice, but otherwise, they are equally likely to arrive at any time. The joint p.d.f. of X, the time Alice arrives (in minutes after 12 P.M.), and Y, the time Bob arrives (in minutes after 12 P.M.), is given

$$f(x,y)=\begin{cases} c & 0\le x<y\le 60 \\ 0 & otherwise \end{cases}$$

where c is a constant.

   a. Sketch a birds-eye view of the joint p.d.f. (Because the p.d.f. is constant, you should be able to calculate all probabilities using geometry, rather than integration).
   b. Determine the value of c that makes this a valid joint p.d.f.
   c. Calculate the probability that Bob arrives more than 25% later than Alice. That is, what is $P(Y>1.25X)$? (Sketch a picture of this region.)
   d. Suppose X and Y are continuous random variables with joint p.d.f. f(x,y). What is P(X=Y) and why?

2. An ecologist selects a point inside a circular sampling region according to a uniform distribution. Let X be the x-coordinate of the point selected and Y be the y-coordinate of the point selected. If the circle is centered at (0,0) and has radius r, then the joint pdf of X and Y is $f(x,y)=\begin{cases} c^2 & x^2+y^2\le r^2 \\ 0 & otherwise \end{cases}$

a) Determine the value of c that makes this a valid joint p.d.f.

b) What is the probability that the selected point is within r/2 of the center of the circular region? (*Hint:* Use geometry.)

c) What is the probability that *both* X and Y differ from 0 by at most r/2?

3. A company produces cans of mixed nuts containing almonds, cashews, and peanuts. Each can is exactly 1 lb, but the amount of each type of nut is random. The joint p.d.f. of X, the amount of almonds, and Y, the amount of cashews, is

$$f(x,y)=\begin{cases} 24xy & 0\le x\le 1, 0\le y\le 1, x+y\le 1, \\ 0 & otherwise \end{cases}$$

Show that the probability there are more almonds than cashews is 0.50.

## .Markov Chain

1. Define (i) Stochastic matrix (ii) Absorbing State (iii) Fixed Probability Vector.
2. Verify that the matrix $A=\begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{pmatrix}$ is a regular stochastic matrix.
3. Verify that the matrix $A=\begin{pmatrix} 0 & 1 \\ 0.3 & 0.7 \end{pmatrix}$ is a regular stochastic matrix.
4. Prove that the Markov Chain whose transition probability matrix is $P=\begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible.
5. Find the fixed probability vector for the transition matrix $P=\begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0 & 0.4 & 0.6 \\ 0 & 0.2 & 0.8 \end{bmatrix}$.
6. Find the unique fixed probability vector for the regular stochastic matrix $A=\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$
.
7. Find the unique fixed probability vector for the regular stochastic matrix $A=\begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}$.
8. Find the unique fixed probability vector for the regular stochastic matrix $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$
.
9. A student's study habits are as follows. If he studies one night, he is 70% sure not to study the next night. On the other hand, if he does not study one night, he is 60% sure not to study the next night. In a long run how often does he study?

10. Consider a game of "ladder climbing". There are 5 levels in the game, level 1 is the lowest (bottom) and level 5 is the highest (top). A player starts at the bottom. Each time, a fair coin is tossed. If it turns up heads, the player moves up one rung. If tails, the player moves down to the very bottom. Once at the top level, the player moves to the very bottom if a tail turns up, and stays at the top if head turns up.
    a) Find the transition probability matrix.
    b) Find the two-step transition probability matrix.
    c) Find the steady-state distribution of the Markov chain.

11. Three boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball .Find the probability after three throws that
    (i) A has the ball (ii) B has the ball (iii) C has the ball

12. The pattern of sunny and rainy days on the planet Rainbow is a homogeneous Markov chain with two states. Every sunny day is followed by another sunny day with probability 0.8. Every rainy day is followed by another rainy day with probability 0.6.
    a) If today is sunny on Rainbow, what is the chance of rain the day after tomorrow?
    b) Compute the probability that April 1 next year is rainy on Rainbow.

13. A computer device can be either in a busy mode (state 1) processing a task, or in an idle mode (state 2), when there are no tasks to process. Being in a busy mode, it can finish a task and enter an idle mode any minute with the probability 0.2. Thus, with the probability 0.8 it stays another minute in a busy mode. Being in an idle mode, it receives a new task any minute with the probability 0.1 and enters a busy mode. Thus, it stays another minute in an idle mode with the probability 0.9. The initial state is idle. Let $X_n$ be the state of the device after n minutes. a) Find the distribution of $X_2$. b) Find the steady-state distribution of $X_n$.

14. An auto insurance company classifies its customers in three categories: poor, satisfactory and preferred. No one moves from poor to preferred or from preferred to poor in one year. 40% of the customers in the poor category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become poor; 20% of those in the preferred category are downgraded to satisfactory.(i) Write the transition matrix for the model (ii) Define Irreducible Markov Chain (iii) Is the Markov chain irreducible?

15. In a certain city, the weather on a day is reported as sunny, cloudy or rainy. If a day is sunny, the probability that the next day is sunny is 70%, cloudy is 20% and rainy is 10%. If a day is cloudy, the probability that the next day is sunny is 30%, cloudy is 20% and rainy is 50%. If a day is rainy, the probability that the next day is sunny is 30%, cloudy is 30% and rainy is 40%. If a Sunday is sunny, find the probability that the Wednesday is rainy.

16. A housewife buys 3 kinds of cereals A, B, and C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys cereal B. However, if she buys cereal B or C, the next week she is three times as likely to buy cereal A as the other cereal. In the long run how often she buys each of the three cereals?

17. A housewife buys 3 brands of soaps: A, B, C. She never buys the same brand on successive weeks. If she buys brand A in a week, she buys brand B in the next week. If she buys the brand other than A in a week, then in the next week she is three times as likely to buy the brand A as the other brand. Supposing that she has brought brand B in the first week, find the probability of her buying each of the three brands in the fourth week.

18. Each year a man trades his car for a new car in 3 brands of the popular company Maruti Udyog Limited. If he has “Standard”, he trades it for “Zen”. If he has a “Zen”, he trades it for“Esteem”. If he has“Esteem”, he is just as likely to trade it for a new “Esteem” or for a “Zen” or a “Standard”.In 2010 he bought his first car which was “Esteem”.Find the probability that he has on 2012 “Esteem”and on 2013 “Zen”.

19. A habitual gambler is a member of two clubs A and B. He visits either of the clubs everyday for playing cards. He never visits club A on two consecutive days. But, if he visits club B on a particular day, then the next day he is as likely to visit club B or club A.
    (i) Find the transition matrix of this Markov chain.
    (ii) Show that the matrix is a regular stochastic matrix.

(iii) Find the unique fixed probability vector.
(iv) If the person had visited club B on Monday, find the probability that he visits club A on Thursday.

20. A gambler's luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However, if he loses a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so
    i. What is the probability of he winning the second game?
    ii. What is the probability of he winning the third game?
    iii. In the long run, how often he will win?

21. Assume that a man's profession can be classified as professional, skilled labourer , or unskilled labourer. Assume that, of the sons of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of sons of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations. Set up the matrix of transition probabilities.
    (i) Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.
    (ii) In the long run, what is the probability that a great grandson of a skilled labourer is a professional man.

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## UNIT-3

## JOINT PROBABILITY & MARKOV CHAIN

1. The joint distribution of two random variables X & Y is given. In each case, find (i) Marginal distribution of X & Y (ii) Co variance & correlation of X & Y.

| Y / X | −2 | −1 | 4 | 5 | f(x) |
|---|---|---|---|---|---|
| 1 | 0.1 | 0.2 | 0 | 0.3 | 0.6 |
| 2 | 0.2 | 0.1 | 0.1 | 0 | 0.4 |
| g(y) | 0.3 | 0.3 | 0.1 | 0.3 | 1 |

ans i) Marginal distribution of x.

| X | 1 | 2 |
|---|---|---|
| f(x) | 0.6 | 0.4 |

Marginal distribution of Y

| Y | −2 | −1 | 4 | 5 |
|---|---|---|---|---|
| g(Y) | 0.3 | 0.3 | 0.1 | 0.3 |

ii) $E(X) = \Sigma x_i f(x_i) = 0.6 + 2(0.4) = 1.4$

$E(Y) = \Sigma y_j \, g(y_j) = -2(0.3) - 1(0.3) + 4(0.1) + 5(0.3)$

$= -0.6 - 0.3 + 0.4 + 1.5$

$= 1$

$E(XY) = \Sigma\Sigma x_i y_j J_{ij} = 1(-2)(0.1) + 1(-1)(0.2) + 1(4)(0)$

$+ 1(5)(0.3) + 2(-2)(0.2) + 2(-1)(0.1)$

$+ 2(4)(0.1) + 2(5)(0)$

$= -0.2 - 0.2 +$ [illegible] $+ 1.5 - \cancel{0.8} - 0.2 + \cancel{0.8}$

$= 0.9$

Covariance $= E(XY) - E(X)E(Y) = 0.9 - 1.4(1) = \underline{\underline{-0.5}}$

Correlation $\rho(X, Y) = \dfrac{cov(X, Y)}{\sigma_x \sigma_y}$

$\sigma_X^2 = E(X^2) - (E(X))^2$

$= 2.2 - (1.4)^2$

$= 0.24$

$\sigma_x = \sqrt{0.24} = 0.4898$

$E(X^2) = \Sigma x_i^2 f(x_i)$

$= 1^2(0.6) + 2^2(0.4)$

$= 0.6 + 1.6$

$= 2.2$

$\sigma_y^2 = E(Y^2) - (E(Y))^2$

$= 10.6 - 1$

$= 9.6$

$\sigma_y = \sqrt{9.6} = 3.0983$

$E(Y^2) = \Sigma y_j^2 g(y_j)$

$= 4(0.3) + (0.3) + 16(0.1) + 25(0.3)$

$= 10.6$

$$\therefore \rho(X,Y) = \frac{-0.5}{(0.4898)(3.0983)} = \underline{-0.3295}$$

2. Two dice are thrown. The random variable X represents the sum of two scores & the random variable Y represents the no. of sixes. Find (i) the joint distribution of X & Y (ii) the individual distribution of X & Y (iii) the probability that X will exceed 7, given Y = 1 (iv) find covariance & correlation between X & Y.

ans $X = \{2, 3, \ldots, 12\}$

$Y = \{0, 1, 2\}$

i)

| X \ Y | 0 | 1 | 2 | f(x) |
|---|---|---|---|---|
| 2 | 1/36 | 0 | 0 | 1/36 |
| 3 | 2/36 | 0 | 0 | 2/36 |
| 4 | 3/36 | 0 | 0 | 3/36 |
| 5 | 4/36 | 0 | 0 | 4/36 |
| 6 | 5/36 | 0 | 0 | 5/36 |
| 7 | 4/36 | 2/36 | 0 | 6/36 |
| 8 | 3/36 | 2/36 | 0 | 5/36 |
| 9 | 2/36 | 2/36 | 0 | 4/36 |
| 10 | 1/36 | 2/36 | 0 | 3/36 |
| 11 | 0 | 2/36 | 0 | 2/36 |
| 12 | 0 | 0 | 1/36 | 1/36 |
| g(y) | 25/36 | 10/36 | 1/36 | 1 |

ii)

| X | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| f(x) | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |

↗ Marginal distribution of X

Marginal distribution of Y

| y | 0 | 1 | 2 |
|---|---|---|---|
| f(y) | 25/36 | 10/36 | 1/36 |

iii) Probability $(x > 7 \,\&\, y = 1) = P(8,1) + P(9,1) + P(10,1) + P(11,1) + P(12,1) = 2/36 + 2/36 + 2/36 + 2/36 + 0$

$= 8/36 = \underline{2/9}$

iv) $E(X) = \sum x_i f(x_i) = 2/36 + 6/36 + 12/36 + 20/36 + 30/36 + 42/36 + 40/36 + 36/36 + 30/36 + 22/36 + 12/36 = 7$

$E(Y) = \sum y_j g(y_j) = 10/36 + 2/36 = 12/36 = 1/3$

$E(XY) = \sum\sum x_i y_j J_{ij} = 7(2/36) + 8(2/36) + 9(2/36) + 10(2/36) + 11(2/36) + 12(2)(1/36) = \frac{14}{36} + \frac{16}{36} + \frac{18}{36} + \frac{20}{36} + \frac{22}{36} + \frac{24}{36}$

$= \frac{19}{6}$

Covariance $= E(XY) - E(X)E(Y) = \frac{19}{6} - 7(1/3) = 5/6 = \underline{0.833}$

Correlation $\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$

$\sigma_x^2 = E(X^2) - (E(X))^2$

$\sigma_x^2 = \frac{329}{6} - (7)^2 = \frac{35}{6}$

$\sigma_x = 2.415$

$\sigma_y^2 = E(Y^2) - (E(Y))^2$

$\sigma_y^2 = \frac{7}{18} - \left(\frac{1}{3}\right)^2 = \frac{5}{18}$

$\sigma_y = 0.527$

$E(X^2) = \sum x_i^2 f(x_i)$

$= \frac{4}{36} + \frac{18}{36} + \frac{48}{36} + \frac{100}{36} + \frac{180}{36} + \frac{294}{36} + \frac{320}{36} + \frac{324}{36} + \frac{300}{36} + \frac{242}{36} + \frac{147}{36}$

$= \frac{329}{6}$

$E(Y^2) = \sum y_j^2 g(y_j)$

$= \frac{10}{36} + 4(1/36) = \frac{7}{18}$

$\therefore \rho(X,Y) = \frac{0.833}{(2.415)(0.527)} = 0.6545$

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3. The distribution of two independent random variables X & Y are given below:

| $X$ | 1 | 2 |
|---|---|---|
| $P(X)$ | 0.7 | 0.3 |

| $Y$ | -2 | 5 | 8 |
|---|---|---|---|
| $P(Y)$ | 0.3 | 0.5 | 0.2 |

Find joint distribution of X & Y. Verify that $Cov(X,Y)=0$.

ans Given X & Y are independent random variable,

$$J_{ij} = P(x_i)\,P(y_j)$$

Joint Distribution Table :-

| X \ Y | -2 | 5 | 8 | f(x) |
|---|---|---|---|---|
| 1 | 0.21 | 0.35 | 0.14 | 0.7 |
| 2 | 0.09 | 0.15 | 0.06 | 0.3 |
| g(y) | 0.3 | 0.5 | 0.2 | 1 |

$$E(X) = \sum x_i\, f(x_i) = 0.7+0.6 = 1.3$$

$$E(Y) = \sum y_j\, g(y_j) = -0.6+2.5+1.6 = 3.5$$

$$E(XY) = \sum\sum J_{ij}\, x_i\, y_j$$

$$= -2(0.21)+5(0.35)+8(0.14)-4(0.09)+10(0.15)+16(0.06)$$

$$= 4.55$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) \Rightarrow 4.55 - 1.3(3.5)$$

$$= \underline{\underline{0}}$$

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4. The joint probability function of two random variables X & y is given by $f(x,y) = c(2x+y)$ where x & y can assume all integral values such that $0 \leq x \leq 2$, $0 \leq y \leq 3$ & $f(x,y)=0$ otherwise. Find

i) The value of constant c

ii) $P(X \geq 1, y < 2)$

iii) Marginal distribution of X & Y

iv) Check whether X & Y are independent

ans Joint Probability Table

| x \ y | 0 | 1 | 2 | 3 | f(x) |
|---|---|---|---|---|---|
| 0 | 0 | C | 2C | 3C | 6C |
| 1 | 2C | 3C | 4C | 5C | 14C |
| 2 | 4C | 5C | 6C | 7C | 22C |
| g(y) | 6C | 9C | 12C | 15C | 1 |

i) $\Sigma f(x) = \Sigma g(y) = 1$

$42C = 1 ; \boxed{C = 1/42}$

ii) $P(x \geq 1, y < 2)$

$= P(1,0) + P(1,1) + P(2,0) + P(2,1)$

$= 2C + 3C + 4C + 5C = 14C = \frac{14}{42} = \underline{\underline{\frac{1}{3}}}$

iii) Marginal distribution of x

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 6/42 | 14/42 | 22/42 |

Marginal distribution of y

| y | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| g(y) | 6/42 | 9/42 | 12/42 | 15/42 |

iv) $E(X) = \Sigma x_i f(x_i)$

$= 0 + \frac{14}{42} + \frac{44}{42} = \frac{58}{42}$

$E(Y) = \Sigma y_j g(y_j)$

$= 0 + \frac{9}{42} + \frac{24}{42} + \frac{45}{42} = \frac{78}{42}$

$E(XY) = \Sigma\Sigma x_i y_j J_{ij}$

$= 1(3C) + 2(4C) + 3(5C) + 2(5C) + 4(6C) + 6(7C)$

$= 3C + 8C + 15C + 10C + 24C + 42C = 102C$

$= 102/42$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{102}{42} - \left(\frac{58}{42}\right)\left(\frac{78}{42}\right) = -0.136 \neq 0$$

∴ $x$ & $y$ are dependent.

5. A coin is tossed 3 times. Let X denote 0 or 1 according as tail or head occurs on the first toss. Let Y denote total no. of tails which occur. Determine (i) marginal distribution of $x$ & Y (ii) Joint probability distribution of X & Y.

ans $S = \{HHH, HHT, HTH, HTT, THT, TTH, THH, TTT\}$

$X = \{0, 1\}$ $\quad$ $Y = \{0, 1, 2, 3\}$

$1^{st} \to T, H$

Joint Probability distribution:

| X ╲ Y | 0 | 1 | 2 | 3 | $f(x)$ |
|---|---|---|---|---|---|
| $1^{st}$ 0 is T | 0 | 1/8 | 2/8 | 1/8 | 4/8 |
| $1^{st}$ 1 is H | 1/8 | 2/8 | 1/8 | 0 | 4/8 |
| $g(y)$ | 1/8 | 3/8 | 3/8 | 1/8 | 1 |

Marginal distribution of X

| X | 0 | 1 |
|---|---|---|
| $f(x)$ | 4/8 | 4/8 |

Marginal distribution of Y

| Y | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $g(Y)$ | 1/8 | 3/8 | 3/8 | 1/8 |

6. Two fruits are selected at random from a bag containing 3 apples, 2 oranges & 4 mangoes. If $x$ & $y$ are respectively the no. of apples & the no. of oranges included among the two fruits drawn from the bag, find the probability associated with all possible pair of values of $x, y$. Also find the correlation between $x$ & $y$.

ans $x \to$ no. of apples drawn, $y \to$ no. of oranges drawn

$X = \{0, 1, 2\}$

$Y = \{0, 1, 2\}$

| X \ Y | 0 | 1 | 2 | $f(x)$ |
|---|---|---|---|---|
| 0 | 6/36 | 8/36 | 1/36 | 15/36 |
| 1 | 12/36 | 6/36 | 0 | 18/36 |
| 2 | 3/36 | 0 | 0 | 3/36 |
| $g(y)$ | 21/36 | 14/36 | 1/36 | 1 |

$P(0,0) = \frac{^4C_2}{^9C_2} = \frac{\frac{4!}{2!2!}}{\frac{9!}{2!7!}} = \frac{6}{36}$

$P(0,1) = \frac{^2C_1 \times {^4C_1}}{^9C_2} =$

$P(0,2) = \frac{^2C_2}{^9C_2} = \frac{1}{36}$

$P(1,0) = \frac{^3C_1 \times {^4C_1}}{^9C_2} = \frac{12}{36}$ $P(1,1) = \frac{^3C_1 \times {^2C_1}}{^9C_2} = \frac{6}{36}$

$P(2,0) = \frac{^3C_2}{^9C_2} = 3/36$

$E(X) = \Sigma x_i f(x_i) = \frac{18}{36} + \frac{6}{36} = \frac{24}{36}$

$E(Y) = \Sigma y_j g(y_j) = \frac{14}{36} + \frac{2}{36} = \frac{16}{36}$

$E(XY) = \Sigma\Sigma x_i y_j J_{ij} = 1(1)\left(\frac{6}{36}\right) = \frac{6}{36}$

$cov(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{6}{36} - \frac{24}{36}\left(\frac{16}{36}\right) = \frac{6}{36}\frac{6-384}{1296} = -0.129$

$\sigma_x^2 = E(X^2) - (E(X))^2$

$= \frac{30}{36} - \left(\frac{24}{36}\right)^2 = 0.388$

$\Rightarrow \sigma_x = 0,622$

$E(X^2) = \Sigma x_i^2 f(x_i)$

$= \frac{18}{36} + 4\left(\frac{3}{36}\right) = \frac{18+12}{36}$

$= \frac{30}{36}$

$E(Y^2) = \Sigma y_j^2 g(y_j)$

$= \frac{14}{36} + \frac{4}{36} = \frac{18}{36}$

$\sigma_y^2 = E(Y^2) - (E(Y))^2$

$= \frac{18}{36} - \left(\frac{16}{36}\right)^2 = 0.3024$

$\Rightarrow \sigma_y = 0.549$

Correlation, $\rho(X,Y) = \frac{cov(X,Y)}{\sigma_x \sigma_y}$

$= \frac{-0.129}{(0.622)(0.549)} = \frac{-0.129}{0.341}$

$\therefore \rho(X,Y) = \underline{\underline{-0.378}}$

7. Two cards are drawn at random from a box which contains 5 cards numbered 1, 1, 2, 2 & 3. Let X denote the sum & Y the maximum of two numbers drawn. Find the joint distribution of X & Y. Also compute cov (X, Y) & $\rho(X, Y)$.

ans $X = \{2, 3, 4, 5\}$ , $Y = \{1, 2, 3\}$

$S = \{ (1^{I}, 1^{II}), (1^{I}, 2^{I}), (2^{I}, 1^{I}), (2^{I}, 2^{II}), (3^{I}, 2^{I}), (2^{I}, 3^{I}), (1^{I}, 3^{I}), (3^{I}, 1^{I}), (1^{II}, 3^{I})$
$(3^{I}, 1^{II}), (1^{II}, 1^{I}), (2^{II}, 2^{I}), (1^{I}, 2^{II}), (2^{II}, 1^{I}), (1^{II}, 2^{I}), (2^{I}, 1^{II}), (1^{II}, 2^{II}), (2^{I}, 1^{I})$
$(2^{II}, 3^{I}), (3^{I}, 2^{II}) \}$

Joint Probability Table:

| X \ Y | 1 | 2 | 3 | f(x) |
|---|---|---|---|---|
| 2 | $2/20$ $^{(1,1)}_{(1,1)}$ | 0 | 0 | $2/20$ |
| 3 | 0 | $8/20$ | 0 | $8/20$ |
| 4 | 0 | $2/20$ | $4/20$ | $6/20$ |
| 5 | 0 | 0 | $4/20$ | $4/20$ |
| g(y) | $2/20$ | $10/20$ | $8/20$ | 1 |

$E(X) = \Sigma x_i f(x_i) = 2(2/20) + 3(8/20) + 4(6/20) + 5(4/20)$

$= \frac{4+24+24+20}{20} = 3.6$

$E(Y) = \Sigma y_j\, g(y_j) = \frac{2}{20} + 2(10/20) + 3(8/20) = 2.3$

$E(XY) = \Sigma\Sigma x_i y_j J_{ij} = \frac{4}{20} + 6(8/20) + 8(2/20) + 12(4/20) + 15(4/20)$

$= 8.8$

$E(X^2) = \Sigma x_i^2 f(x_i) = 4(2/20) + 9(8/20) + 16(6/20) + 25(4/20)$

$= \frac{276}{20} = 13.8$

$E(Y^2) = \Sigma y_j^2\, g(y_j) = \frac{2}{20} + 4(10/20) + 9(8/20) = \frac{114}{20}$

$\sigma_x^2 = E(X^2) - (E(X))^2 = \frac{276}{20} - \left(\frac{72}{20}\right)^2 = 0.84 \Rightarrow \underline{\sigma_x = 0.9165}$

$\sigma_y^2 = E(Y^2) - (E(Y))^2 = \frac{114}{20} - (2.3)^2 = 0.41 \Rightarrow \underline{\sigma_y = 0.6403}$

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= 8.8 - (3.6)(2.3)$$

$$= \underline{\underline{0.52}}$$

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_x \sigma_y} = \frac{0.52}{(0.9165)(0.6403)} = \underline{\underline{0.886}}$$

17/12/24

## ☆ Markov Chain:-

1. A student's study habits are as follows: If he studies one day, he is 70% sure not to study the next day. On the otherhand, if he does not study one day he is 60% sure not to study the next day. In long run, how often will he study?

ans State space = {S, N}    S→Study
N→Not study

Transition Probability matrix: $A = \begin{array}{c|cc} & S & N \\ \hline S & 0.3 & 0.7 \\ N & 0.4 & 0.6 \end{array}$ tomorrow

Today

$1$
$-0.3$

For long run problems:

VA = V (where V is probability vector)
where V = [x y] & x+y = 1

$$[x \quad y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \quad y]$$

$$0.3x + 0.4y = x$$
$$0.7x + 0.6y = y$$
$$x + y = 1$$

Two variables & 3 eqⁿs so choose any one with x+y=1.

$$\begin{aligned} 0.3x + 0.4y &= x \\ x + y &= 1 \end{aligned} \Rightarrow x = \frac{4}{11} \ \& \ y = \frac{7}{11}$$

$$V = \left[\frac{4}{11} \quad \frac{7}{11}\right]$$

∴ Probability that he studies in long run = $\frac{4}{11}$ //

2. Professor Symons either walks to school, or he rides his bicycle. If he walks to school one day, then the next day, he will walk or cycle with equal probability. But if he bicycles one day, then the probability that he will walk the next day is 1/4. Express this information in a transition matrix.

i) if it is assumed that the initial day is Monday, write a matrix that gives probabilities of a transition from Monday to Wednesday.

ii) In the long run, how often will he walk to school, & how often will he bicycle?

W→ walk to school, B→ bicycle

ans $S = \{W, B\}$

$$\text{TPM: } A = \begin{array}{c|cc} & W & B \\ \hline W & 0.5 & 0.5 \\ B & 0.25 & 0.75 \end{array} \text{ Tomorrow}$$

Today

i) ans Mon — Tue — Wed

$$\text{Mon} \xrightarrow{A} \text{Tue} \xrightarrow{A^2} \text{Wed}$$

$$A^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0.375 & 0.625 \\ 0.3125 & 0.6875 \end{bmatrix}}}$$

ii) ans Let $V = [x \quad y]$ be unique fixed probability vector such that

$$VA = V \; \& \; x + y = 1$$

$$[x \quad y] \begin{bmatrix} 0.5 & 0.5 \\ 0.25 & 0.75 \end{bmatrix} = [x \quad y]$$

$$0.5x + 0.25y = x$$

$$x + y = 1$$

$$x = \frac{1}{3} \; \& \; y = \frac{2}{3}$$

∴ In the long run, probability that he will walk is $\frac{1}{3}$ & probability that he will bicycle is $\underline{\underline{\frac{2}{3}}}$.

3. In the Dark Ages, Harvard, Dartmouth & Yale admitted only male students. Assume that, at that time, 80% of the sons of Harvard men went to Harvard & the rest went to Yale, 40% of sons of Yale men went to Yale, & the rest split evenly between Harvard & Dartmouth; & of the sons of Dartmouth men, 70% went to Dartmouth, 20% to Harvard & 10% to Yale.

i) Find the probability that the grandson of a man from Harvard went to Harvard

ii) Modify the above by assuming that the son of a Harvard man always went to Harvard. Again, find the probability that the grandson of a man from Harvard went to Harvard.

ans $S = \{H, D, Y\}$

$H \to$ Harvard
$D \to$ Dartmouth
$Y \to$ Yale

$$\text{TPM: } A = \begin{array}{c} \\ H \\ D \\ Y \end{array}\begin{array}{c} \begin{array}{ccc} H & D & Y \end{array} \\ \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \end{array} \quad 2^{nd} \text{ gen.}$$

$1^{st}$ gen.

Initial Probab. vector
$P^{(0)} = (1 \ 0 \ 0)$

P - S - GS
$P^{(0)}$ $P^{(1)}$ $P^{(2)}$

$P^{(1)} = P^{(0)}A$

$P^{(2)} = P^{(1)}A$

$= (P^{(0)}A)A = P^{(0)}A^2$

i) ans $p^{(2)} = p^{(0)}A^2$

$$= [1 \quad 0 \quad 0]\begin{bmatrix} 0.7 & 0.06 & 0.24 \\ 0.33 & 0.52 & 0.15 \\ 0.42 & 0.33 & 0.25 \end{bmatrix}$$

$$= (0.7 \quad 0.06 \quad 0.24)$$

∴ Probability that the grandson went to Harvard is 70% or 0.7

ii) ans $B = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$

$$p^{(2)} = p^{(0)} B^2$$

$$= [1 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 & 0 \\ 0.37 & 0.52 & 0.11 \\ 0.48 & 0.33 & 0.19 \end{bmatrix}$$

$$= [1 \quad 0 \quad 0]$$

∴ Probability that grandson of man from Harvard went to Harvard is 1.

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4. A city is served by two cable TV companies, BestTV & CableCast. Due to their aggressive sales tactics, each year 40% of BestTV customers switch to CableCast; the other 60% of BestTV customers stay with BestTV. On the other hand, 30% of CableCast customers switch to BestTV. The two states in this example are BestTV & CableCast. Express the information above as a transition matrix which displays the probabilities of going from one state into another state.

i) Suppose that today 1/4 of customers subscribe to BestTV & 3/4 of customers subscribe to CableCast. After 1 year, what percent subscribe to each company?

ans $S = \{B, C\}$

TPM: $P = \begin{array}{c|cc} & B & C \\ \hline B & 0.6 & 0.4 \\ C & 0.3 & 0.7 \end{array}$ Next year

This year

1) ans $p^{(0)} = (1/4 \quad 3/4)$

$p^{(1)} = p^{(0)} P$

$$\begin{bmatrix} 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.375 & 0.625 \end{bmatrix}$$

∴ 37.5% subscribe to Best TV

62.5% subscribe to CableCast

---

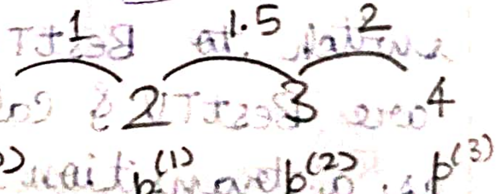
5\. In analysing switching by Business class customers between airlines the following data has been obtained by British Airways (BA).

$$\begin{bmatrix} 0.85 & 0.15 \\ 0.1 & 0.9 \end{bmatrix}$$

For example if the last flight by a Business Class customer by BA, the probability that their next flight is by BA is 0.85. Business Class customers make 2 flights a year on average. Currently BA has 30% of Business Class market. What would you forecast BA's share of the Business class market ~~to~~ after two years?

ans $S = \{B, O\}$ $B \to BA$, $O \to$ Others

$$TPM : P = \begin{array}{c|cc} & B & O \\ \hline B & 0.85 & 0.15 \\ O & 0.1 & 0.9 \end{array}$$



$p^{(0)} = [0.3 \quad 0.7]$

$p^{(3)} = p^{(0)} P^3$ (∵ 2 flights/year)

$$= [0.3 \quad 0.7] \begin{bmatrix} 0.653125 & 0.3468 \\ 0.2312 & 0.7687 \end{bmatrix}$$

$$= [0.3577 \quad 0.6421]$$

∴

6\. Verify that the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$ is a regular stochastic matrix

ans $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

∴ It is not a regular stochastic matrix as it did not change

7\. Verify that the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$ is a regular

ans $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4444 & 0.4444 & 0.1111 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4444 & 0.4444 & 0.4444 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4814 & 0.4814 & 0.037 \end{bmatrix}$

∴ This is not a regular stochastic matrix, as the no. of zeroes does not reduce.

8\. Verify if $A = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{bmatrix}$ is a regular stochastic matrix

ans $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0.125 & 0.3125 & 0.5625 \\ 0.5 & 0.25 & 0.25 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0.125 & 0.3125 & 0.5625 \\ 0.5 & 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.25 & 0.25 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.1562 & 0.6406 & 0.2031 \\ 0.125 & 0.3125 & 0.5625 \end{bmatrix}$

∴ This is a regular stochastic matrix, as the no. of zeroes are reducing (as every entry in mass → discrete, density → continuous

$A^3$ is non zero, A is a regular stochastic matrix).

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## ☆ Joint Probability : Continuous Case :-

If x & y are two continuous random variables, then if $f(x,y)$, a real valued function satisfying the conditions $f(x,y) \geq 0$ & $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ is called as joint probability function or joint density function.

**NOTE:** $P(a \leq x \leq b,\ c \leq y \leq d) = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx.$

→ Marginal distribution of x

$F_1(x) =$ ~~$\int_{u=-\infty}^{x}\int_{v=-\infty}^{\infty} f(u,v)\,dv\,du = P(X \leq x)$~~ $= \int_{-\infty}^{\infty} f(x,y)\,dy$

→ Marginal distribution of y

$F_2(y) =$ ~~$P(Y \leq y) = \int_{u=-\infty}^{\infty}\int_{v=-\infty}^{y} f(u,v)\,dv\,du$~~ $= \int_{-\infty}^{\infty} f(x,y)\,dx$

→ Expectation of X:

$$E(X) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} x\, f(x,y)\,dy\,dx$$

→ Expectation of Y:

$$E(Y) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} y\, f(x,y)\,dy\,dx$$

→ Covariance of X & Y

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$E(XY) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} xy\, f(x,y)\,dy\,dx$$

→ X & Y are independent random variable if $Cov(X,Y) = 0$ or $E(XY) = E(X)E(Y)$

→ $\sigma_x^2$ = Variance of $X = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (X - \mu_x)^2 f(x,y)\,dx\,dy$

→ $\sigma_y^2$ = Variance of $Y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (Y-\mu_y)^2 f(x,y)\,dx\,dy$

★ Problems:

1. Let $x$ & $y$ be random variables having the joint density function

$$f(x,y) = \begin{cases} 4xy, & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal density functions of $x$ & $y$. Verify that $E(x+y) = E(x) + E(y)$ & $E(xy) = E(x)E(y)$

ans Marginal distribution of $X$

$F_1(x) =$

$\int_{y=-\infty}^{\infty} f(x,y)dy$

$= \int_{y=0}^{1} 4xy\,dy$

$= 4x\frac{y^2}{2}\Big]_0^1$

$= \underline{\underline{2x}}$

(Remaining integrals, func$^n$ value is 0 (from $-\infty$ to 0)

So we have considered $\left(\int_{-\infty}^{0}\right)$ × $\int_{0}^{1}$ ✓ $\left(\int_{1}^{\infty}\right)$ ×

Marginal distribution of $Y$

$F_2(y) =$

$= \int_{x=-\infty}^{\infty} f(x,y)dx$

$= \int_{x=0}^{1} 4xy\,dx$

$= 4y\frac{x^2}{2}\Big]_0^1$

$= \underline{\underline{2y}}$

$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dy\,dx$

$= \int_{x=0}^{1}\int_{y=0}^{1} x\, 4xy\,dy\,dx = 4\int_{x=0}^{1} \frac{x^2y^2}{2}\Big]_0^1 dx$

$= 2\frac{x^3}{3}\Big]_0^1 = \underline{\underline{\frac{2}{3}}}$

$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 4xy^2\, dy\, dx = 4 \int_{x=0}^{1} \left[\frac{xy^3}{3}\right]_0^1 dx$$

$$= \frac{4}{3} \left.\frac{x^2}{2}\right]_0^1 = \underline{\underline{\frac{2}{3}}}$$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx$$

$$= \int_{0}^{1} \int_{0}^{1} 4x^2y^2\, dy\, dx = 4 \int_{0}^{1} \left[x^2 \frac{y^3}{3}\right]_0^1 dx$$

$$= \frac{4}{3} \left.\frac{x^3}{3}\right]_0^1 = \underline{\underline{\frac{4}{9}}}$$

$$E(X+Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 4xy(x+y)\, dy\, dx = \int_{x=0}^{1} \int_{y=0}^{1} 4x^2y + 4xy^2\, dy\, dx$$

$$= 4 \int_{x=0}^{1} \int_{y=0}^{1} x^2y + xy^2\, dy\, dx = 4 \int_{x=0}^{1} \left[x^2\frac{y^2}{2} + \frac{xy^3}{3}\right]_0^1 dx$$

$$= 4 \int_{x=0}^{1} \left(\frac{x^2}{2} + \frac{x}{3}\right) dx = 4 \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1 = 4 \times \frac{2}{6} = \underline{\underline{\frac{4}{3}}}$$

$E(X+Y) = \frac{4}{3}$; $E(X) + E(Y) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ $\Rightarrow \therefore \underline{E(X+Y) = E(X) + E(Y)}$

$E(X)E(Y) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ ; $E(XY) = \frac{4}{9}$ $\Rightarrow \therefore \underline{E(XY) = E(X)E(Y)}$

2. Suppose the joint P.D.F. of $(x, y)$ is given by

$$p(x,y) = \begin{cases} e^{-y}, & x > 0,\ y > x \\ 0 & \text{elsewhere} \end{cases}$$

Find (i) the marginal density function of $x$
(iii) the marginal density function of $y$.
Check whether $x$ & $y$ are independent.

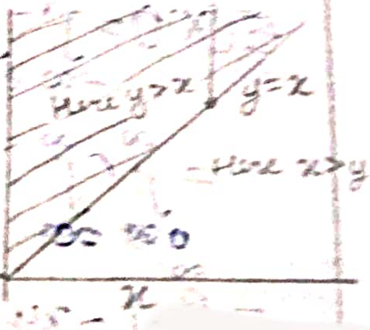
ans **Marginal dist. of X**

$F_1(x) =$ [redacted] $\int_{y=-\infty}^{\infty} f(x,y)\,dy$

$= \int_{y=x}^{\infty} e^{-y}\,dy$

$= -[e^{-\infty} - e^{-x}]$

$= \underline{\underline{e^{-x}}}$



[redacted]

$$E(X) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x\, f(x,y)\,dy\,dx$$

$$= \int_{x=0}^{\infty} \int_{y=x}^{\infty} x e^{-y}\,dy\,dx = \int_{x=0}^{\infty} \left[\frac{x e^{-y}}{-1}\right]_{y=x}^{\infty} dx = -\int_{x=0}^{\infty} [-x e^{-x}]\,dx$$

$$= + \left[x \frac{e^{-x}}{-1} - \int \left(\frac{d}{dx} x \int e^{-x} dx\right) dx\right]_{x=0}^{\infty} = +\left[-x e^{-x} - \int (-e^{-x})\,dx\right]_{x=0}^{\infty}$$

$$= \left[-x e^{-x} - e^{-x}\right]_0^{\infty} = -[0 - 1] = \underline{\underline{1}}$$

$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy = \int_{x=0}^{\infty} \int_{y=x}^{\infty} y e^{-y}\,dy\,dx$$

$$= \int_{x=0}^{\infty} \left[y \frac{e^{-y}}{-1} - \int \frac{e^{-y}}{-1} \cdot 1\,dy\right] dx = \int_{x=0}^{\infty} \left[-y e^{-y} - e^{-y}\right]_x^{\infty} dx$$

$$= \int_{x=0}^{\infty} \cancel{x+1} \left[x e^{-x} + e^{-x}\right] dx = \int_{x=0}^{\infty} (x+1) e^{-x}\,dx$$

$$= \left[(x+1) \frac{e^{-x}}{-1} - \int \frac{e^{-x}}{-1}\,dx\right]_{x=0}^{\infty} = \left[-(x+1)e^{-x} - e^{-x}\right]_{x=0}^{\infty}$$

$$= 1 + 1 = \underline{2}$$

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx$$

$$=\int_{0\cancel{\infty}}^{\infty}\int_{\cancel{x}\infty}^{\infty} x\,y\,e^{-y}\,dy\,dx=\int_{0\cancel{\infty}}^{\infty}\left[x\left(y\frac{e^{-y}}{-1}-\int\frac{e^{-y}}{-1}dy\right)\right]_{\cancel{\infty}}^{\infty}dx$$

$$=\int_{0\cancel{\infty}}^{\infty}\left[-xy\underbrace{e^{-y}}_{u}-xe^{-y}\right]_{x\cancel{\infty}}^{\infty}dx=\int_{0\cancel{\infty}}^{\infty}\left[x^{2}\cancel{y}e^{-x}+xe^{-x}\right]dx$$

Bernoulli's rule

$$=\int_{0}^{\infty}e^{-x}(x^2+x)\,dx=\left[(x^2+x)\frac{e^{-x}}{-1}-(2x+1)\frac{e^{-x}}{+1}+\frac{2e^{-x}}{-1}\right]_0^\infty$$

$$u\int v-\frac{d}{dx}u\int v$$

$$=1+2=3 \qquad 0+1+2=0$$

$$E(XY)=3\neq E(X)E(Y)$$

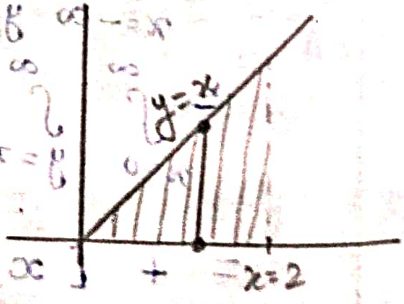
$\therefore$ X & Y are dependent random variables

3. The joint probability function for $(x,y)$ is given by

$$f(x,y)=\begin{cases}cxy, & 0\le x\le 2,\ 0\le y\le x\\ 0 & \text{otherwise}\end{cases}$$

Determine c & hence evaluate $P(\frac{1}{2}<x<1)$.

ans W.K.T. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx=1$



$$\int_{x=0}^{2}\int_{y=0}^{x} cxy\,dy\,dx=1$$

$$c\int_{x=0}^{2}\left.x\frac{y^2}{2}\right]_0^x dx=1 \Rightarrow \frac{c}{2}\int_{x=0}^{2}\left[x^3\right]dx=1$$

$$\frac{c}{2}\left.\frac{x^4}{4}\right]_0^2=1 \Rightarrow \frac{c}{8}\times 2^4$$

$$\Rightarrow \cancel{\tfrac{2}{2}}\, 2c=1\Rightarrow c=\frac{1}{2}$$

$\iint$ denotes volume

$$P\left(\frac{1}{2}<x<1\right)=\int_{x=1/2}^{1}\int_{y=0}^{x} cxy\,dy\,dx$$

$$=\frac{1}{2}\int^{1}\left.x.\frac{y^2}{2}\right]^{x} dx$$

$$= \frac{1}{4}\int_{x=1/2}^{1} x^3\,dx = \frac{1}{4}\times\frac{x^4}{4}\Big]_{1/2}^{1} = \frac{1}{16}\left[1-\left(\tfrac{1}{2}\right)^4\right] = \frac{1}{16}\left[1-\frac{1}{16}\right]$$

$$= \frac{1}{16}\times\frac{15}{16} = \underline{\underline{\frac{15}{256}}}$$

48. For the distribution given by the density function

$$f(x,y) = \begin{cases} \frac{1}{96}xy, & 0<x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

Evaluate: (i) $P(1<x<2,\ 2<y<3)$

(ii) $P(x \geq 3,\ y \leq 2)$ (iii) $P(x+y \leq 3)$

~~(iv) P(x + y ≤~~

ans W.K.T. $P(a \leq x \leq b,\ c \leq y \leq d) = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$

i) $$P(1<x<2,\ 2<y<3) = \int_{x=1}^{2}\int_{y=2}^{3} \frac{1}{96}xy\,dy\,dx = \frac{1}{96}\int_{x=1}^{2} x\frac{y^2}{2}\Big]_{y=2}^{3} dx$$

$$= \frac{1}{96\times 2}\int_{x=1}^{2}\left[x(3)^2 - x(2)^2\right]dx = \frac{1}{96\times 2}\int_{x=1}^{2}\left[9x-4x\right]dx$$

$$= \frac{5}{96\times 2}\ \frac{x^2}{2}\Big]_{1}^{2} = \frac{5}{96\times 2}\left[\frac{4}{2}-\frac{1}{2}\right] = \frac{5}{96\times 2}\times\frac{3}{2}$$

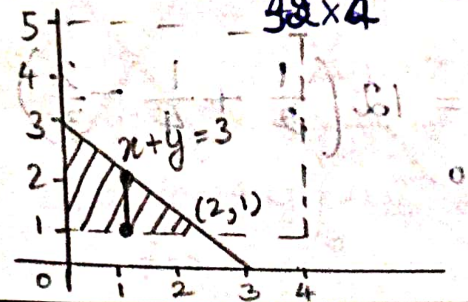
$$= \cancel{\frac{45}{768}}\quad \frac{15}{384} = \underline{\underline{\frac{5}{128}}}$$

ii) $$P(x \geq 3,\ y \leq 2) = \int_{x=3}^{4}\int_{y=1}^{2} \frac{1}{96}xy\,dy\,dx = \frac{1}{96}\int_{x=3}^{4} x\frac{y^2}{2}\Big]_{y=1}^{2} dx$$

$$= \frac{1}{96\times 2}\int_{x=3}^{4} 3x\,dx = \frac{3}{96\times 2}\int_{x=3}^{4} x\,dx = \frac{1}{32\times 2}\ \frac{x^2}{2}\Big]_{3}^{4}$$

$$= \frac{1}{32\times 4}(16-9) = \frac{7}{32\times 4} = \underline{\underline{\frac{7}{128}}}$$

iii)



$$P(x+y\leq 3) = \int_{x=0}^{2}\int_{y=1}^{3-x}\frac{1}{96}xy\,dy\,dx = \frac{1}{96}\int_{x=0}^{2}\left.x\frac{y^2}{2}\right]_{1}^{3-x}dx$$

$$= \frac{1}{96\times 2}\int_{x=0}^{2}\left[x\overset{9+x^2-6x}{(3-x)^2}-x\right]dx = \frac{1}{96\times 2}\int_{x=0}^{2}\left[9x+x^3-6x^2-x\right]dx$$

$$= \frac{1}{96\times 2}\int_{x=0}^{2}\left[8x+x^3-6x^2\right]dx = \frac{1}{96\times 2}\left[\frac{\overset{4}{\cancel{8}}x^2}{\cancel{2}_1}+\frac{x^4}{4}-\frac{\overset{2}{\cancel{6}}x^3}{1\cancel{3}}\right]_{x=0}^{2}$$

$$= \frac{1}{96\times 2}\left[4(4)+\frac{\overset{4}{\cancel{16}}}{\cancel{4}}-2(8)\right] = \frac{1}{96\times 2}\left[\cancel{16}+4-\cancel{16}\right] = \frac{2}{96}$$

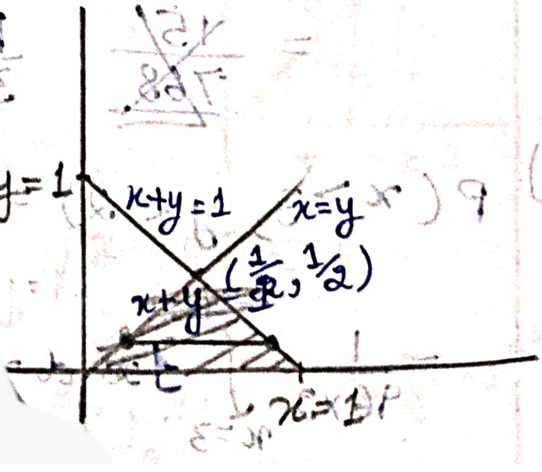
$$= \frac{1}{48}$$

5\. A company produces cans of mixed nuts containing almonds, cashews & peanuts. Each can is exactly 1 lb, but the amount of each type of nut is random. The joint p.d.f. of X, the amount of almonds, & Y, the amount of cashews, is

$$f(x,y) = \begin{cases} 24xy & 0\leq x\leq 1,\ 0\leq y\leq 1,\ x+y\leq 1, \\ 0 & \text{otherwise} \end{cases}$$

Show that the probability that there are more almonds than cashews is 0.50.

ans To prove that : $P(X > Y) = 0.5$



$$P(X>Y) = \int_{y=0}^{1/2}\int_{x=y}^{1-y} 24xy \, dx\, dy = 24\int_{y=0}^{1/2} \left[\frac{x^2 y}{2}\right]_{x=y}^{1-y} dy$$

$$= 12\int_{y=0}^{1/2} \left(y(1-y)^2 - y^3\right) dy = 12\int_{y=0}^{1/2} [y(1+y^2-2y)-y^3]dy$$

$$= 12\int_{y=0}^{1/2} [y+\cancel{y^3}-2y^2-\cancel{y^3}]dy = 12\left[\frac{y^2}{2}-\frac{2y^3}{3}\right]_{y=0}^{1/2}$$

$$= 12\left[\frac{(1/4)}{2}-\frac{2(1/8)}{3}\right] = 12\left[\frac{1}{8}-\frac{2}{24}\right]$$

$$= 12\left[\frac{1}{8}-\frac{1}{12}\right] = 12\left(\frac{1}{24}\right) = \underline{\underline{0.5}}$$

Hence Proved.

6. An ecologist selects a point inside a circular sampling region according to a uniform distribution. Let X be the X-co-ordinate of the point selected & Y be the Y-co-ordinate of the point selected. If the circle is centered at (0, 0) & has radius, r, then the joint pdf of X & Y is $f(x,y) = \begin{cases} c^2 & x^2+y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases}$

i) Determine value of c that makes this a valid joint pdf.

ii) What is the probability that the selected point is within r/2 of the center of the circular region?

iii) What is the probability that both X & Y differ from 0 by at most r/2?

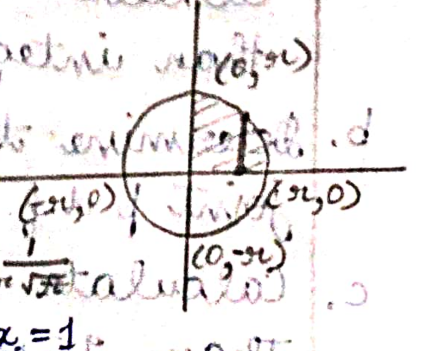
ans i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1 \Rightarrow$

or

Area of cylinder = 1

$\pi r^2 \times f(x,y) = 1$

$\pi r^2 \times c^2 = 1 \Rightarrow c = \frac{1}{\sqrt{\pi}}$



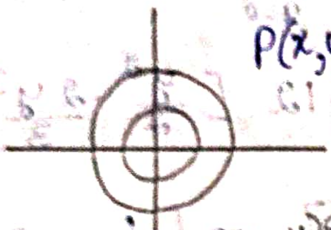
$$4\int_{x=0}^{r}\int_{y=0}^{\sqrt{r^2-x^2}} c^2 \, dy\, dx = 1$$

$$4c^2\int_{x=0}^{r} y\Big]_0^{\sqrt{r^2-x^2}} dx = 1 \Rightarrow 4c^2\int_{x=0}^{r} \sqrt{r^2-x^2}\, dx = 1$$

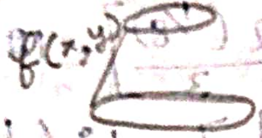
$$\Rightarrow 4c^2\left[\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r=1$$

$$\Rightarrow 4c^2\left[\frac{r^2}{2}\times\frac{\pi}{2}\right]=1 \Rightarrow 4c^2\left[\frac{r^2\pi}{4}\right]=1 \Rightarrow c^2=\frac{1}{\pi r^2} \Rightarrow c=\frac{1}{r\sqrt{\pi}}$$

ii)



$P(x,y)=$ Volume of cylinder with radius $r/2$ & height $f(x,y)=\pi.\frac{r^2}{4}\times c^2$



$$=\frac{\cancel{\pi r^2}}{4}\times\frac{1}{\cancel{\pi r^2}}$$

$$\therefore P(x,y)=\frac{1}{4}$$

iii) Same as ii)

7. Alice & Bob meet for lunch every day at a random time between noon & 1 P.M. Bob always arrives after Alice, but otherwise, they are equally likely to arrive at any time. The joint p.d.f of X, the time Alice arrives (in minutes after 12 P.M.), & Y, the time Bob arrives (in minutes after 12P.M.) is given by

$$f(x,y)=\begin{cases} c & 0\le x<y\le 60 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

a. Sketch a birds-eye view of the joint p.d.f (Because the p.d.f. is constant, you should be able to calculate all probabilities using geometry, rather than integration).

b. Determine the value of c that makes this a valid joint p.d.f.

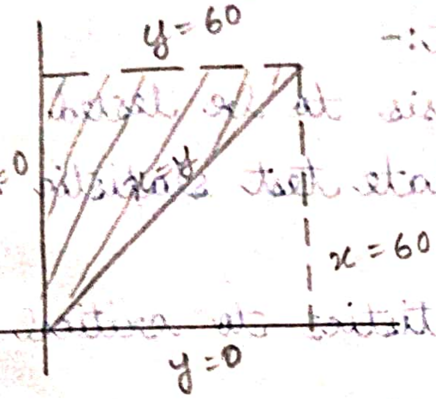
c. Calculate the probability that Bob arrives more than 25% later than Alice. That is what is

$P(Y > 1.25X)$ ?

d. Suppose X & Y are continuous random variables with joint p.d.f. $f(x, y)$. What is $P(X = Y)$ & why?

Ans:-

a.



b.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) = 1 \Rightarrow \int_{x=0}^{60}\int_{y=x}^{60} c\, dy\, dx = 1 \Rightarrow c\int_{x=0}^{60} y\Big]_{x}^{60} dx = 1$$

$$\Rightarrow c\int_{x=0}^{60} (60 - x)\, dx = 1 \Rightarrow c\left[60x - \frac{x^2}{2}\right]_0^{60} = 1$$

$$\Rightarrow c\left[3600 - \frac{3600}{2}\right] = 1 \Rightarrow c\left[1800\right] = 1 \Rightarrow c = \frac{1}{1800}$$

Or Volume of region = area of region × h

$$1 = \frac{1}{2} \times 60 \times 60 \times f(x, y)$$

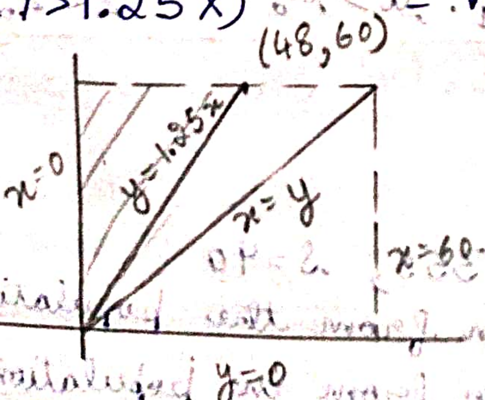
$$1 = 30 \times 60 \times c \Rightarrow c = \frac{1}{1800}$$

c. $P(Y > 1.25X)$ = Volume of shaded region

= area of shaded region × height

$$= \frac{1}{2} \times 48 \times 60 \times c$$

$$= 48 \times 30 \times \frac{1}{1800} = \frac{48}{60} = 0.8.$$



d.