B.M.S. COLLEGE OF ENGINEERING, BENGALURU – 560 0 19
Autonomous college, affiliated to VTU
DEPARTMENT OF MATHEMATICS

STATISTICS AND DISCRETE MATHEMATICS

(Course Code: 23MA3BSSDM)

UNIT-5: COMBINATORICS

I. Binomial Theorem and multinomial theorem:

1. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$.
2. Find the coefficient of

(i) x^9y^3 in the expansion of $(2x-3y)^{12}$

(ii) x^0 in the expansion of $\left(3x^2-\frac{2}{x}\right)^{15}$

(iii) x^{12} in the expansion of $x^3(1-2x)^{10}$

3. Determine the coefficient of (i) xyz^2 in the expansion of $(2x-y-z)^4$

(ii) $x^2y^2z^3$ in the expansion of $(3x-2y-4z)^7$

(iii) $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

II. The Principle of Inclusion Exclusion and Generalizations of The Principles:

1. Out of 1200 students at a college, 582 took economics, 627 took English, 543 to mathematics 217 took both economics and English, 307 took both economics and mathematics, 250 took both mathematics and English and 212 took all three courses. How many took none of the three?
2. Out of 30 students in a hospital, 15 study History, 8 study Economics and 6 study Geography. It is known that 3 students study all these subjects. Show that 7 or more students' study none of these subjects.
3. Among the first 500 positive integers, determine the integers which are not divisible by 2, nor by 3 nor by 5.
4. Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study biology and physics and 30 do not study any of the subjects. Find the number of students studying all three subjects?
5. How many integers between 1 and 300 (inclusive) are

   a) Divisible by at least one of 5, 6, 8?

   b) Divisible by none of 5, 6, 8?
6. In how many ways 5 number of a's, 4 number of b's and 3 number of c's can be arranged so that all the identical letters are not in a single block?

7. In how many ways can the 26 letters of the English alphabet be permuted so that none of the pattern's CAR, DOG, PUN or BYTE occurs?
8. Find the number of permutations of the letters *a,b,c ...x,y,z* in which none of the patterns *spin, game, path* or *net* occurs.

III. Catalan Numbers:

1. Define Catalan number. Obtain the number of paths from $(2,1)$ to $(7,6)$ and not rise above the line $y = x - 1$ using the moves $R: (x, y) \to (x + 1, y)$ and $U: (x, y) \to (x, y + 1)$.
2. Using the moves $R:(x,y) \to (x+1,y)$ and $U:(x,y) \to (x,y+1)$. Find in how many ways can one go from $(2,6)$ to $(6,10)$ and not rise above the line $y = x + 4$.
3. Using the moves $R: (x, y) \to (x + 1, y)$ and $U: (x, y) \to (x, y + 1)$. Find in how many ways can one go from $(7,3)$ to $(10,6)$ and not rise above the line $y = x - 4$.
4. Using the moves $R: (x, y) \to (x + 1, y)$ and $U: (x, y) \to (x, y + 1)$. Find in how many ways can one go from $(3,8)$ to $(11,16)$ and not rise above the line $y = x + 5$.

IV. Derangements :(Nothing is in its Right Place)

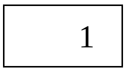
1. Find the number of derangements of 1,2,3,4.
2. Evaluate d_5, d_6, d_7, d_8.
3. While at race track, a person bets on each of the ten horses in a race to come in accordance to how they are favoured. In how many ways can they reach the finish line so that he losses all his bets?
4. For the positive integers 1, 2, 3,…, n there are 11660 derangements where 1,2,3,4,5 appear in the first five positions. What is the value of 'n'?
5. In how many ways can the integers 1, 2, 3,..,10 be arranged in a line so that no even integer is in its natural place?
6. There are *n* pairs of children's gloves in a box. Each pair is of a different colour. Suppose the right gloves are distributed at random to *n* children, and thereafter the left gloves are also distributed to them at random. Find the probability that
   (i) No child gets a matching pair,
   (ii) Every child gets a matching pair,
   (iii) Exactly one child gets a matching pair,
   (iv) At least 2 children get matching pairs.
7. Thirty students take a quiz. Then for the purpose of grading, the teacher asks the students to exchange papers. Find the probability that
   (i) No one is grading his own paper.
   (ii) Every student gets his own paper.
   (iii) Exactly one student gets his own paper.
8. At a restaurant, 10 men hand over their umbrellas to the receptionist. In how many ways can their umbrellas be returned so that
   (i) No man receives his own umbrella?

(ii) At least one of the men receives his own umbrella?
(iii) At least two of the men receives his own umbrella?

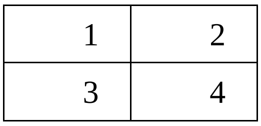
V. Rook Polynomials:

1. Find the Rook polynomial for the following boards for the non-shaded squares.

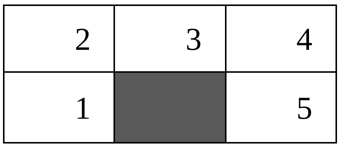
a.



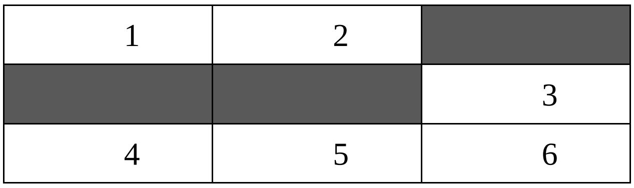
b.



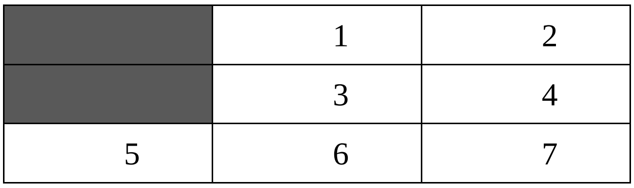
c.



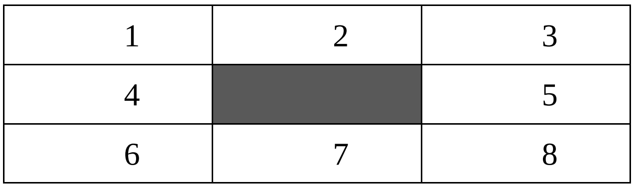
d.



e.

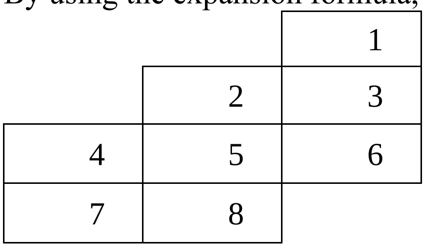


f.

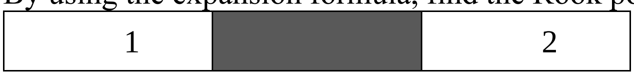


2. Find the Rook polynomial for the 2×2 board by using the expansion formula.

3. By using the expansion formula, find the Rook polynomial for the board 'C' shown below.



4. By using the expansion formula, find the Rook polynomial for the board 'C' shown below.



3	4	5
	6	

5. Find the Rook polynomial for the board ‘C’ shown below.

1	2			
3	4			
			5	6
			7	8
		9	10	11

6. Find the Rook polynomial for the board ‘C’ shown below.

	1	2		
3		4		
	5		6	7
			8	

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5.1 The Rules of Sum and Product

In many situations of computational work, we employ two basic rules of counting, called the *Sum Rule* and the *Product Rule*. These rules are stated and illustrated in the following paragraphs.

The Sum Rule

Suppose two tasks T_1 and T_2 are to be performed. If the task T_1 can be preformed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 or T_2) can be performed in $m + n$ ways.

More generally, if $T_1, T_2, T_3, \ldots T_k$ are k tasks such that no two of these tasks can be performed at the same time and if the task T_i can be performed in n_i different ways, then one of the k tasks (namely T_1 or T_2 or $T_3 \ldots$, or T_k) can be performed in $n_1 + n_2 + \ldots + n_k$ different ways.

Example 1. Suppose there are 16 boys and 18 girls in a class and we wish to select one of these students (either a boy or a girl) as the class representative. The number of ways of selecting a boy is 16 and the number of ways of selecting a girl is 18. Therefore, the number of ways of selecting a student (boy or girl) is $16 + 18 = 34$.

Example 2. Suppose a Hostel library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Science and 11 books on Electronics. Suppose a student wishes to choose one of these books for study. The number of ways in which he can choose a book is $12 + 10 + 16 + 11 = 49$.

Example 3 . Suppose T_1 is the task of selecting a prime number less than 10 and T_2 is the task of selecting an even number less than 10. Then T_1 can be performed in 4 ways (– by selecting 2 or 3 or 5 or 7), and T_2 can be performed in 4 ways (– by selecting 2 or 4 or 6 or 8). But, since 2 is both a prime and an even number less than 10, the task T_1 or T_2 can be performed in $4 + 4 - 1 = 7$ ways.

The Product Rule

Suppose that two tasks T_1 and T_2 are to be performed one after the other. If T_1 can be performed in N_1 different ways, and for each of these ways T_2 can be performed in n_2 different ways, then both of the tasks can be performed in $n_1 n_2$ different ways.

More generally, suppose that k tasks $T_1, T_2, T_3, \ldots, T_k$ are to be performed in a sequence. If T_1 can be performed in n_1 different ways and for each of these ways T_2 can be performed in n_2 different ways, and for each of $n_1 n_2$ different ways of performing T_1 and T_2 in that order, T_3 can be performed in n_3 different ways, and so on, then the sequence of tasks $T_1, T_2, T_3, \ldots, T_k$ can be performed in $n_1 n_2 n_3 \ldots n_k$ different ways.

Example 4 . Suppose a person has 8 shirts and 5 ties. Then he has $8 \times 5 = 40$ different ways of choosing a shirt and a tie.

Example 5 . Suppose we wish to construct sequences of four symbols in which the first 2 are English letters and the next 2 are single digit numbers. If no letter or digit can be repeated, then the number of different sequences that we can construct is $26 \times 25 \times 10 \times 9 = 58500$.* If repetition of letters and digits is allowed then the number of different sequences that we can construct is $26 \times 26 \times 10 \times 10 = 67600$.

Example 6 . Suppose a restaurant sells 6 South Indian dishes, 4 North Indian dishes, 3 hot beverages and 2 cold beverages. For breakfast, a student wishes to buy 1 South Indian dish and 1 hot beverage, or 1 North Indian dish and 1 cold beverage. Then he can have the first choice in $6 \times 3 = 18$ ways and he can have the second choice in $4 \times 2 = 8$ ways. The total number of ways he can buy his breakfast items is $18 + 8 = 26$**.

5.2 Permutations

Suppose that we are given n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the first object, and, after this is done, $n - 1$ ways of choosing the second object,..., and finally $n - r + 1$ ways of choosing the r^{th} object, it follows by the Product Rule of counting (stated in the preceding section) that the number of different arrangements, or ***permutations*** (as they are commonly called) is $n(n-1)(n-2)\cdots(n-r+1)$. We denote this number by $P(n, r)$* and is referred to as the *number of permutations of size r of n objects*. Thus (by definition),

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1).$$

*Some other notations for $P(n, r)$ are: ${}^nP_r$, ${}_nP_r$ and $P_{n,r}$.

Using the *factorial notation* defined by

$$k! = k(k-1)(k-2)\cdots 2\cdot 1,$$

for any positive integer k, and $0! = 1$, we find that

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

As a particular case of this, we get

$$P(n,n) = n!.$$

That is, the number of different arrangements (permutations) of n distinct objects, taken all at a time, is $n!$. This is simply called the *number of permutations of n distinct objects.*

In the above analysis, we have considered the situation where all the objects that are to be arranged are *distinct*.

A generalization

Suppose it is required to find the number of permutatuions that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of a second type,..., n_k are of k^{th} type, with $n_1 + n_2 + \cdots + n_k = n$. Then, the number of permutations of the n objects (taken all of them at a time) is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Proof: There are $n!$ permutations when all the n objects are different. We must therefore divide $n!$ by $n_1!$ to account for the fact that the n_1 objects which are alike will identify $n_1!$ of these permutations (for any given set of positions of the n_1 objects in the permutation). Similarly, we must divide $n!$ by $n_2!, n_3!, \ldots, n_k!$, which are the numbers of permutations of the corresponding alike objects. Thus, $n!$ divided by all of $n_1!, n_2!, \ldots, n_k!$ gives the required number of permutations. ∎

Example 1 *In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the other by rotation?*

▶ Let one of them be seated anywhere. Then the remaining $n - 1$ persons can be seated in $(n-1)!$ ways. This is the total number of ways of arranging the n persons in a circle. ∎

Example 2 *It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?*

▶ The 5 men may be seated in odd places in 5! ways and the 4 women may be seated in even places in 4! ways, and corresponding to each arrangement of the men there is an arrangement of the women. Therefore, the total number of arrangements of the desired type is

$$5! \times 4! = 120 \times 24 = 2880.$$

■

Example 3 *In how many ways can 6 men and 6 women be seated in a row*

(i) *if any person may sit next to any other?*

(ii) *if men and women must occupy alternate seats?*

▶ (i) If any person may sit next to any other, no distinction need be made between men and women in their seating. Accordingly, since there are 12 persons in all, the number of ways they can be seated is

$$12! = 479{,}001{,}600.$$

(ii) When men and women are to occupy alternate seats, the six men can be seated in 6! ways in odd places and the six women can be seated in 6! ways in even places, and corresponding to each arrangement of the men there is an arrangement of the women. Therefore, the number of ways in which the men occupy the odd places and the women the even places is

$$6! \times 6! = 720 \times 720 = 518400.$$

Similarly, the number of ways in which the women occupy the odd places and the men the even places is 518400. Accordingly, the total number of ways is

$$518400 + 518400 = 1{,}036{,}800.$$

■

Example 4 *Four different mathematics books, five different computer science books and two different control theory books are to be arranged in a shelf. How many different arrangements are possible if* (a) *the books in each particular subject must all be together?* (b) *only the mathematics books must be together?*

▶ (a) The mathematics books can be arranged among themselves in 4! different ways, the computer science books in 5! ways, the control theory books in 2! ways, and the three groups in 3! ways. Therefore the number of possible arrangements is

$$4! \times 5! \times 2! \times 3! = 24 \times 120 \times 2 \times 6 = 34{,}560.$$

5.3 Combinations

Suppose we are interested in selecting (choosing) a *set* of r objects from a set of $n \geq r$ objects without regard to order. The set of r objects being selected is traditionally called a *combination of r objects* (or briefly *r-combination*).

The total number of combinations of r different objects that can be selected from n different objects can be obtained by proceeding in the following way. Suppose this number is equal to C, say; that is, suppose there is a total of C number of combinations of r different objects chosen from n different objects. Take any one of these combinations. The r objects in this combination can be arranged in $r!$ different ways. Since there are C combinations, the total number of permutations is $C \cdot r!$. But this is equal to $P(n, r)$. Thus,

$$C \cdot r! = P(n, r), \quad \text{or} \quad C = \frac{P(n, r)}{r!}.$$

Thus, the total number of combinations of r different objects that can be selected from n different objects is equal to $P(n, r)/r!$. This number is denoted by $C(n, r)$, or $\binom{n}{r}$*. Thus,

$$C(n, r) \equiv \binom{n}{r} \equiv \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!} \quad \text{for} \quad 0 \leq r \leq n.$$

Replacing r by $n - r$ in this expression, we get

$$C(n, n-r) = \binom{n}{n-r} = \frac{n!}{r!(n-r)!} = C(n, r) = \binom{n}{r} \quad \text{for} \quad 0 \leq r \leq n.$$

Consequently, we have

$$C(n, n) = \binom{n}{n} = C(n, 0) = \binom{n}{0} = 1 \quad \text{and} \quad C(n, 1) = \binom{n}{1} = C(n, n-1) = \binom{n}{n-1} = n.$$

Note: For $r > n$, $C(n, r) \equiv \binom{n}{r}$ is *defined* to be equal to zero.

Example 1 *How many committees of 5 with a given chairperson can be selected from 12 persons?*

▶ The chairperson can be chosen in 12 ways, and, following this, the other four on the committee can be chosen in $C(11, 4)$ ways. Therefore, the possible number of such committees is

$$12 \times C(11, 4) = 12 \times \frac{11!}{4!\,7!} = 12 \times 330 = 3960.$$ ∎

*Some other notations for this number are ${}_nC_r$, ${}^nC_r$ and $C_{n,r}$.

5.3.1 Binomial and Multinomial Theorems

One of the basic properties of $C(n, r) \equiv \binom{n}{r}$ is that it is the coefficient of $x^r y^{n-r}$ and $x^{n-r} y^r$ in the expansion of the expression $(x + y)^n$, where x and y are any real numbers. In other words,

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

This result is known as the ***Binomial Theorem for a positive integral index.***

The numbers $\binom{n}{r}$ for $r = 0, 1, 2, \ldots n$, namely $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$, in the above result are called the *binomial coefficients*.

The student is already familiar with the proof by mathematical induction of the above-mentioned binomial theorem.

The following is a generalization of the binomial theorem, known as the ***Multinomial Theorem***.

<u>**Theorem***</u> **:** *For positive integers n and k, the coefficient of* $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_k^{n_k}$ *in the expansion of* $(x_1 + x_2 + \cdots + x_k)^n$ *is*

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

where each n_i *is a nonnegative integer* $\leq n$, *and* $n_1 + n_2 + n_3 + \cdots + n_k = n$.

<u>Proof:</u> We note that in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ the coefficient of $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ remaining factors, and so on. Therefore, this coefficient is, by the product rule,

$$C(n, n_1) \cdot C(n - n_1, n_2) \cdot C(n - n_1 - n_2, n_3) \cdots\cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$$

$$= \frac{n!}{n_1!\,(n - n_1)!} \cdot \frac{(n - n_1)!}{n_2!\,(n - n_1 - n_2)!} \cdot \frac{(n - n_1 - n_2)!}{n_3!(n - n_1 - n_2 - n_3)!} \cdots \frac{(n - n_1 - n_2 - \cdots - n_{k-1})!}{n_k!\,(n - n_1 - n_2 - \cdots - n_{k-1} - n_k)!}$$

$$= \frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

This proves the required result. •

Note: (1) Another way of stating the Multinomial Theorem is:

When n is a positive integer, the general term in the expansion of

$$(x_1 + x_2 + x_3 + \cdots + x_k)^n \quad \text{is} \quad \frac{n!}{n_1!\, n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $n_1, n_2, \ldots n_k$ *are nonnegative integers not exceeding n and* $n_1 + n_2 + n_3 + \cdots + n_k = n$.

The expression $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$ is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_k}$$

and is called a ***multinomial coefficient***.

*The case $k = 2$ of this Theorem corresponds to the Binomial theorem for a positive integral index.

Example 4 *Find the coefficient of*

(i) x^9y^3 *in the expansion of* $(2x - 3y)^{12}$.

(ii) x^0 *in the expansion of* $\left(3x^2 - \frac{2}{x}\right)^{15}$.

(iii) x^{12} *in the expansion of* $x^3(1 - 2x)^{10}$.

(iv) x^k *in the expansion of* $(1 + x + x^2)(1 + x)^n$, *where n is a positive integer and* $0 \le k \le n + 2$.

▶ (i) We have, by the binomial theorem

$$(2x - 3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} \cdot (2x)^r(-3y)^{12-r}$$
$$= \sum_{r=0}^{12} \binom{12}{r} 2^r(-3)^{12-r} \cdot x^r y^{12-r}$$

In this expansion, the coefficient of x^9y^3 (which corresponds to $r = 9$) is

$$\binom{12}{9} 2^9(-3)^3 = -(2^9 \times 3^3) \times \frac{12!}{9!\,3!} = -2^9 \times 3^3 \times \frac{12 \times 11 \times 10}{6}$$
$$= -2^{10} \times 3^3 \times 11 \times 10$$

(ii) By Binomial theorem, we have

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} \binom{15}{r} (3x^2)^r \left(-\frac{2}{x}\right)^{(15-r)}$$
$$= \sum_{r=0}^{15} \binom{15}{r} 3^r(-2)^{(15-r)} x^{3r-15}$$

The coefficient of x^0 (namely the constant term) which corresponds to $r = 5$ in this is

$$\binom{15}{5} \times 3^5 \times (-2)^{10} = \frac{15!}{10!\,5!} \times 3^5 \times 2^{10}.$$

(iii) By Binomial theorem, we have

$$(1 - 2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} 1^{10-r}(-2x)^r$$

Therefore,

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} (-2)^r x^{r+3}$$

The coefficient of x^{12} in this expansion (which corresponds to $r = 9$) is

$$\binom{10}{9}(-2)^9 = -(10 \times 2^9) = -5120.$$

(iv) Using Binomial theorem, we find that

$$(1+x+x^2)(1+x)^n = (1+x+x^2) \times \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} x^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r + \sum_{r=0}^{n} \binom{n}{r} x^{r+1} + \sum_{r=0}^{n} \binom{n}{r} x^{r+2}$$

The coefficient of x^k in this is

$$\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-2}$$

<u>Note:</u> For example, the coefficients of x^7 and x^{18} are

$$\binom{n}{7} + \binom{n}{6} + \binom{n}{5} \quad \text{and} \quad \binom{n}{18} + \binom{n}{17} + \binom{n}{16}$$

respectively. ∎

Example 5 *Determine the coefficient of*

(i) xyz^2 *in the expansion of* $(2x - y - z)^4$

(ii) $x^2y^2z^3$ *in the expansion of* $(3x - 2y - 4z)^7$.

(iii) $x^{11}y^4$ *in the expansion of* $(2x^3 - 3xy^2 + z^2)^6$

(iv) $a^2b^3c^2d^5$ *in the expansion of* $(a + 2b - 3c + 2d + 5)^{16}$.

▶ (i) By the Multinomial theorem, we note that the general term in the expansion of $(2x-y-z)^4$ is

$$\binom{4}{n_1,\ n_2,\ n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}.$$

For $n_1 = 1, n_2 = 1$ and $n_3 = 2$, this becomes

$$\binom{4}{1,\ 1,\ 2}(2x)(-y)(-z)^2 = \binom{4}{1,\ 1,\ 2} \times 2 \times (-1) \times (-1)^2 xyz^2$$

This shows that the required coefficient is

$$\binom{4}{1, 1, 2} \times 2 \times (-1) \times (-1)^2 = \frac{4}{1!\ 1!\ 2!} \times (-2)$$

$$= -12.$$

(ii) The general term in the expansion of $(3x - 2y - 4z)^7$ is

$$\binom{7}{n_1,\ n_2,\ n_3}(3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$$

For $n_1 = 2, n_2 = 2, n_3 = 3$, this becomes

$$\binom{7}{2,\ 2,\ 3}(3x)^2(-2y)^2(-4z)^3$$

$$= \binom{7}{2,\ 2,\ 3}\left\{3^2 \times (-2)^2 \times (-4)^3\right\} x^2y^2z^3$$

This shows that the required coefficient is

$$\left\{3^2 \times (-2)^2 \times (-4)^3\right\} \times \binom{7}{2,\ 2,\ 3} = -(9 \times 4 \times 64) \times \frac{7!}{2!\ 2!\ 3!}$$

$$= -2304 \times \frac{7 \times 6 \times 5 \times 4}{4}$$

$$= -4{,}83{,}840.$$

(iii) The general term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is

$$\binom{6}{n_1,\ n_2,\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

For $n_3 = 0, n_2 = 2, n_1 = 3$ this becomes

$$\binom{6}{3,\ 2,\ 0}(2^3x^9)(3^2x^2y^4) = 72 \times \frac{6!}{3!\ 2!\ 0!}x^{11}y^4.$$

Thus, the required coefficient is

$$72 \times \frac{6 \times 5 \times 4}{2} = 4320.$$

(iv) The general term in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$ is

$$\binom{16}{n_1,\ n_2,\ n_3,\ n_4,\ n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}.$$

For $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n_4 = 5$ and $n_5 = 16 - (2 + 3 + 2 + 5) = 4$, this becomes

$$\binom{16}{2, 3, 2, 5, 4}a^2(2b)^3(-3c)^2(2d)^5 5^4 = \binom{16}{2, 3, 2, 5, 4} \times 2^3 \times (-3)^2 \times 2^5 \times 5^4 \times a^2b^3c^2d^5$$

Therefore, the coefficient of $a^2b^3c^2d^5$ is

$$2^8 \times 3^2 \times 5^4 \times \frac{16!}{2!\ 3!\ 2!\ 5!\ 4!} = 3 \times 2^5 \times 5^3 \times \frac{16!}{(4!)^2}$$ ■

Exercises

5.4 Catalan Numbers

Consider the sequence $b_0, b_1, b_2, \ldots$ of positive integers, defined by

$$b_0 = 1 \quad \text{and} \quad b_n = \frac{C(2n, n)}{n+1} \equiv \frac{1}{(n+1)} \cdot \frac{(2n)!}{(n!)^2} \quad \text{for} \quad n \geq 1.$$

This sequence is called the ***Catalan Sequence*** and the terms of the sequence are called the **Catalan numbers.**

From the expression for b_n, we find that

$$b_1 = \frac{1}{2}C(2,1) = 1,$$

$$b_2 = \frac{1}{3}C(4,2) = \frac{1}{3} \cdot \frac{4!}{2!\,2!} = 2,$$

$$b_3 = \frac{1}{4}C(6,3) = \frac{1}{4} \cdot \frac{6!}{3!\,3!} = 5,$$

$$b_4 = \frac{1}{5}C(8,4) = \frac{1}{5} \cdot \frac{8!}{4!\,4!} = 14,$$

$$b_5 = \frac{1}{6}C(10,5) = \frac{1}{6} \cdot \frac{10!}{5!\,5!} = 42, \quad \text{and so on.}$$

Thus,

$$b_0 = 1,\ b_1 = 1,\ b_2 = 2,\ b_3 = 5,\ b_4 = 14,\ b_5 = 42$$

are the first six Catalan numbers.

Note: The expression for b_n for $n \geq 1$ given above, namely $b_n = \dfrac{C(2n,n)}{n+1}$ can be put in the following alternative form:

$$b_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

Proof:

$$\binom{2n}{n} - \binom{2n}{n-1} = C(2n,n) - C(2n,n-1)$$

$$= \frac{(2n)!}{n!(2n-n)!} - \frac{(2n)!}{(n-1)!\{2n-(n-1)\}!} = \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!(n+1)!}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!(n)}{(n+1)(n!)(n!)} = \frac{(2n)!}{n!\,n!}\left(1 - \frac{n}{n+1}\right) = C(2n,n) \cdot \frac{n+1-n}{n+1}$$

$$= C(2n,n) \cdot \frac{1}{n+1} = b_n.$$

Catalan numbers arise in many situations. A few of these situations are described below:

Situation 1. Consider the xy-plane with O as the origin and P as a point with (n, n) as its coordinates, where n is a positive integer. In this plane, suppose we wish to reach the point P by starting from the origin O and by making only the following two kinds of moves R and U:

$$R : (x, y) \rightarrow (x+1, y), \quad U : (x, y) \rightarrow (x, y+1).$$

Evidently, from a chosen point, R is a horizontal move of one unit to the right and U is a vertical move of one unit in the upward direction.

Suppose there is a restriction that, in our movement, we may touch the line $y = x$ but never rise above it.

A path (route) in which we move from O to P under the stated restrictions is called a *good path* from O to P.

Evidently, a good path from O to P is an arrangement of n number of $R's$ and n number of $U's$. The number of such paths happens to be the Catalan number b_n.

Thus, for $n \geq 1$, *the Catalan number b_n represents the number of good paths from the origin O to the point $P(n, n)$, where n is a positive integer.*

For example, suppose $P = (3, 3)$. Then the arrangement $RURURU$ is a good path from O to P. This is shown by thick lines in Figure 5.1. Let us denote this path by p_1; that is, $p_1 : RURURU$.

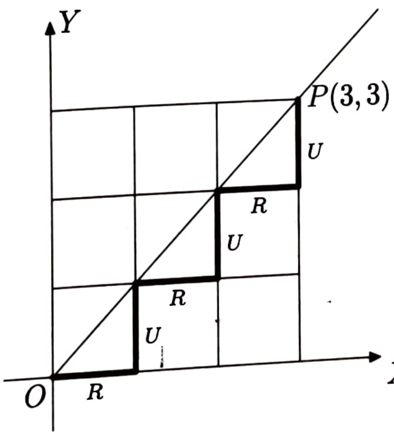

Figure 5.1

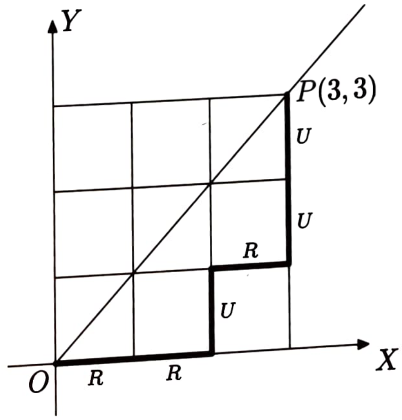

Figure 5.2

The other good paths from O to P are:

$p_2 : RRURUU$ (See Figure 5.2)

$p_3 : RRUURU, \quad p_4 : RURRUU, \quad p_5 : RRRUUU.$

Thus, the number of good paths from O to $P(3, 3)$ is 5, and $5 = b_3$.

Similarly, the number of good paths from O to $P(4, 4)$ is found to be $14 = b_4$.*

Example : *Using the moves*

$$R(x, y) \to (x + 1, y) \quad \textit{and} \quad U : (x, y) \to (x, y + 1)$$

find in how many ways can one go

*The student may identify all these paths.

(a) *from* $(0,0)$ *to* $(6,6)$ *and not rise above the line* $y = x$?

(b) *from* $(2,1)$ *to* $(7,6)$ *and not rise above the line* $y = x - 1$?

(c) *from* $(3,3)$ *to* $(10,15)$ *and not rise above the line* $y = x + 5$?

(7,12)

▶ (a) The required number of ways is 6. Using the definition of b_n, we find that

$$b_6 = \frac{1}{7}C(12,6) = \frac{1}{7}\cdot\frac{12!}{6!\,6!} = 132.$$

(b) Let us shift the origin to (2, 1) and set $X = x - 2$, $Y = y - 1$. Then (2, 1) is the origin for the (X, Y) coordinate system (new) and the given point $(7,6)$ becomes $(5,5)$ in the (X, Y) system. Also, the given line $y = x - 1$ becomes the line $Y = X$ in the (X, Y) system. Accordingly, the required number here is

$$b_5 = \frac{C(10,5)}{6} = \frac{1}{6}\cdot\frac{10!}{5!\,5!} = 42.$$

(c) Shifting the origin to $(3,3)$, we find, as in the above case, that the required number is

$$b_7 = \frac{1}{8}C(14,7) = \frac{1}{8}\cdot\frac{14!}{7!\,7!} = 429.$$ ∎

6.1 The Principle of Inclusion-Exclusion

Recall that if S is a finite set, then the number of elements in S is called the *order* (or the *size*, or the *cardinality*) of S and is denoted by $|S|$. If A and B are subsets of S, then the order of $A \cup B$ is given by the formula*.

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1}$$

Thus, for determining the number of elements that are in $A \cup B$, we *include* all elements in A and B, but *exclude* all elements common to A and B.

Noting that $\overline{A} \cap \overline{B} = (\overline{A \cup B})$ and $|(\overline{A \cup B})| = |S| - |A \cup B|$, we find, by using formula (1), that

$$|\overline{A} \cap \overline{B}| = |(\overline{A \cup B})| = |S| - |A \cup B| = |S| - |A| - |B| + |A \cap B| \tag{2}$$

The formulas (1) and (2) are equivalent to one another, and either of these is referred to as the *Addition Principle* or the *Principle of inclusion-exclusion*, for two sets.

Below we prove a generalization of this principle to n sets.

Principle of Inclusion-Exclusion for *n* sets

Let S be a finite set and $A_1, A_2, \ldots, A_n$ be subsets of S. Then the *Principle of inclusion-exclusion* for $A_1, A_2, \ldots, A_n$ states that

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| + \cdots$$
$$+ (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| \qquad (3)^{**}$$

Example 1 *Among the students in a hostel, 12 students study Mathematics (A), 20 study Physics (B), 20 study Chemistry (C), and 8 study Biology (D). There are 5 students for A and B, 7 students for A and C, 4 students for A and D, 16 students for B and C, 4 students for B and D, and 3 students for C and D. There are 3 students for A, B and C, 2 for A, B, and D, 2 for B, C and D, 3 for A, C and D. Finally, there are 2 who study all of these subjects. Furthermore, there are 71 students who do not study any of these subjects. Find the total number of students in the hostel.*

▶ From, what is given, we have

$$|A| = 12, \quad |B| = 20, \quad |C| = 20, \quad |D| = 8,$$
$$|A \cap B| = 5, \quad |A \cap C| = 7, \quad |A \cap D| = 4,$$
$$|B \cap C| = 16, \quad |B \cap D| = 4, \quad |C \cap D| = 3,$$

$$|A \cap B \cap C| = 3, \quad |A \cap B \cap D| = 2, \quad |B \cap C \cap D| = 2,$$
$$|A \cap C \cap D| = 3, \quad |A \cap B \cap C \cap D| = 2, \quad |\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}| = 71,$$

in the notation which is obvious.

We are required to find $|S|$ where S is the set of all students in the hostel.

The Principle of inclusion-exclusion (as given by equation (4)) applied to the above data gives

$$71 = |S| - (12 + 20 + 20 + 8) + (5 + 7 + 4 + 16 + 4 + 3) - (3 + 2 + 2 + 3) + 2 = |S| - 29$$

This gives $$|S| = 71 + 29 = 100.$$

Thus, the total number of students in the hostel is 100. ■

Example 2 *Out of* 30 *students in a hostel,* 15 *study History,* 8 *study Economics, and* 6 *study Geography. It is known that* 3 *students study all these subjects. Show that* 7 *or more students study none of these subjects.*

▶ Let S denote the set of all students in the hostel, and A_1, A_2, A_3 denote the sets of students who study History, Economics and Geography, respectively. Then, from what is given, we have

$$S_1 = \Sigma|A_i| = 15 + 8 + 6 = 29, \quad \text{and} \quad S_3 = |A_1 \cap A_2 \cap A_3| = 3.$$

The number of students who do not study any of the three subjects is $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$. This is given by (see expression (4))

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_1 \cap A_2 \cap A_3| \\ &= |S| - S_1 + S_2 - S_3 \\ &= 30 - 29 + S_2 - 3 = S_2 - 2 \qquad ((i)) \end{aligned}$$

where $S_2 = \Sigma|A_i \cap A_j|$.

We note that $(A_1 \cap A_2 \cap A_3)$ is a subset of $(A_i \cap A_j)$ for $i, j = 1, 2, 3$. Therefore, *each* of $|A_i \cap A_j|$, which are 3 in number, is greater than or equal to $|A_1 \cap A_2 \cap A_3|$. Hence

$$S_2 = \Sigma|A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3| = 9.$$

Using this in (i), we find that

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| \geq 9 - 2 = 7.$$

This proves the required result ■

Example 4 *How many integers between* 1 *and* 300 *(inclusive) are*

(i) *divisible by at least one of* 5, 6, 8?

(ii) *divisible by none of* 5, 6, 8?

▶ Let $S = \{1, 2, \ldots, 300\}$ so that $|S| = 300$. Also, let A_1, A_2, A_3 be subsets of S whose elements are divisible by 5, 6, 8 respectively. Then:

(i) The number of elements of S that are divisible by at least one of 5, 6, 8 is $|A_1 \cup A_2 \cup A_3|$. This is given by (see equation (3))

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} + |A_1 \cap A_2 \cap A_3| \quad \text{(i)}$$

We note that

$$|A_1| = \lfloor 300/5 \rfloor = 60, \quad |A_2| = \lfloor 300/6 \rfloor = 50, \quad |A_3| = \lfloor 300/8 \rfloor = 37,$$
$$|A_1 \cap A_2| = \lfloor 300/30 \rfloor = 10, \quad |A_1 \cap A_3| = \lfloor 300/40 \rfloor = 7,$$
$$|A_2 \cap A_3| = \lfloor 300/24 \rfloor = 12 \quad \text{(Note that the l.c.m. of 6 and 8 is 24)},$$
$$|A_1 \cap A_2 \cap A_3| = \lfloor 300/120 \rfloor = 2 \text{ (Note that the l.c.m. of 5, 6, 8 is 120)}.$$

Using these in (i), we get

$$|A_1 \cup A_2 \cup A_3| = (60 + 50 + 37) - (10 + 7 + 12) + 2 = 120.$$

Thus, 120 elements of S are divisible by at least one of 5, 6, 8.

(ii) The number of elements of S that are divisible by none of 5, 6, 8 is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - |A_1 \cup A_2 \cup A_3| = 300 - 120 = 180.$$ ■

Example 8 *In how many ways 5 number of a's, 4 number of b's and 3 number of c's can be arranged so that all the identical letters are not in a single block?*

► The given letters are 5 + 4 + 3 = 12 in number, of which 5 are a's 4 are b's and 3 are c's. If S is the set of all permutations (arrangements) of these letters, we have

$$|S| = \frac{12!}{5!\,4!\,3!}.$$

Let A_1 be the set of arrangements of the letters where the 5 a's are in a single block. The number of such arrangements is

$$|A_1| = \frac{8!}{4!\,3!}.$$

(Because in such an arrangement all the a's taken together can be regarded as a single letter and the remaining letters consist of 4 b's and 3 c's).

Similarly, if A_2 is the set of arrangements where the 4 b's are in a single block, and A_3 is the set of arrangements where the 3 c's are in a single block, we have

$$|A_2| = \frac{9!}{5!\,3!} \quad \text{and} \quad |A_3| = \frac{10!}{5!\,4!}.$$

Likewise,

$$|A_1 \cap A_2| = \frac{5!}{3!}, \quad |A_1 \cap A_2| = \frac{6!}{4!}, \quad |A_2 \cap A_3| = \frac{7!}{5!}, \quad |A_1 \cap A_2 \cap A_3| = 3!.$$

Accordingly, the required number of arrangements is

$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| \\
&\qquad + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3| \\
&= \frac{12!}{5!\,4!\,3!} - \left\{\frac{8!}{4!\,3!} + \frac{9!}{5!\,3!} + \frac{10!}{5!\,4!}\right\} + \left\{\frac{5!}{3!} + \frac{6!}{4!} + \frac{7!}{5!}\right\} - 3! \\
&= 27720 - (280 + 504 + 1260) + (20 + 30 + 42) - 6 \\
&= 25762.
\end{aligned}$$

∎

$$+ (20! + 0 + 0 + 0 + 0 + 21!) - (0 + 0 + 0 + 0) + 0$$
$$= 26! - \{3 \times (23!) + 24!\} + (20! + 21!).$$ ■

Example 11 *In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns* CAR, DOG, PUN *or* BYTE *occurs?*

▶ Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$.

Let A_1 be the set of all permutations in which CAR appears. This word, CAR, consists of three letters which form a single block. The set A_1 therefore consists of all permutations which contain this single block and the 23 remaining letters. Therefore, $|A_1| = 24!$

Similarly, if A_2, A_3, A_4 are the sets of all permutations which contain DOG, PUN and BYTE respectively, we have

$$|A_2| = 24!, \quad |A_3| = 24!, \quad |A_4| = 23!.$$

**If there is a permutation containing both of the patterns *spin* and *path*, this permutation contains a repeated letter, *p*.

†Note that the pattern *spinet* includes in it both of the patterns *spin* and *net*.

Likewise, we find that*

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26 - 6 + 2)! = 22!,$$

$$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26 - 7 + 2) = 21!,$$

$$|A_1 \cap A_2 \cap A_3| = (26 - 9 + 3)! = 20!,$$

$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = (26 - 10 + 3)! = 19!,$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26 - 13 + 4)! = 17!.$$

Therefore, the required number of permutations is given by

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$
$$= 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!) - (20! + 3 \times 19!) + 17! \qquad \blacksquare$$

6.2 Derangements

A permutation of n distinct objects in which *none* of the objects is in its natural (original) place is called a ***derangement***. For example, a permutation of the integers $1, 2, 3, 4, \ldots n$, in which 1 is not in the first place, 2 is not in the second place, 3 is not in the third place, and so on, and n is not in the nth place is a derangement.

The number of possible derangements of n distinct objects $1, 2, 3, \ldots, n$ is denoted by d_n. If there is only one object, it continues to be in its original place in every arrangement; therefore $d_1 = 0$. If there are two objects, a derangement can be done in only one way – by interchanging their places; therefore $d_2 = 1$. For three objects 1, 2, 3, the possible derangements are 231 and 312; therefore $d_3 = 2$.

Formula for d_n

The following is the formula for d_n for $n \geq 1$:

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!} \qquad (1)$$

$|A_1 \cap A_2| = (n-2)!$. Similarly, the order of the intersection of every two of the sets $A_1, A_2, \ldots$ is $(n-2)!$. The number of such intersections is $C(n, 2)$.

Accordingly,

$$S_2 = \Sigma|A_i \cap A_j| = C(n, 2) \times (n-2)!.$$

Likewise, we find

$$S_3 = \Sigma|A_i \cap A_j \cap A_k| = C(n, 3) \times (n-3)!$$

$$\ldots\ldots$$

$$S_n = |A_1 \cap A_2 \cap \ldots \cap A_n| = C(n, n) \times (n-n)! = C(n, n).$$

Using these in expression (2), we get

$$d_n = (n!) - C(n, 1) \times (n-1)! + C(n, 2) \times (n-2)! - C(n, 3) \times (n-3)! + \cdots + (-1)^n C(n, n)$$

$$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!}$$

$$= n! \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\} = n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

This completes the proof.*

Remark: Recall the exponential expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

From this, we get

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

To five places of decimals, e^{-1} is known to be equal to 0.36788. Thus,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 0.36788 \quad \text{to five places.}$$

On the other hand, we find that

$$\sum_{k=0}^{7} \frac{(-1)^k}{k!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots - \frac{1}{7!} \approx 0.36786.$$

*For another proof, see Example 19, Section 6.3

Example 1 *Find the number of derangements of* 1, 2, 3, 4.

▶ Here, there are 4 objects. Therefore, the number of derangements is

$$d_4 = 4! \times \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\}$$

$$= 24 \times \left\{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right\}$$

$$= 12 - 4 + 1 = 9.$$

We can check that the nine derangements of 1, 2, 3, 4 are:

2143	2341	2413
3142	3412	3421
4123	4312	4321

■

Example 2 *Evaluate* d_5, d_6, d_7, d_8.

▶ We have

$$d_5 = (5!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$$

$$= (120)\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44,$$

$$d_6 = (6!)\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right\}$$

$$= (720)\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right) = 265,$$

$$d_7 \approx \lfloor (7!) \times e^{-1} \rfloor \approx \lfloor 5040 \times 0.3679 \rfloor \approx 1854,$$

$$d_8 \approx \lfloor (8!) \times e^{-1} \rfloor \approx \lfloor 40320 \times 0.3679 \rfloor \approx 14833.$$

■

Example 3 *While at the race track, a person bets on each of the nine horses in a race to come in accordance to how they are favoured. In how many ways can they reach the finish line so that he loses all his bets?*

▶ Here, we have to find the number of ways of arranging the horses $1, 2, 3, \ldots, 9$ so that 1 is not in its favoured place, 2 is not in its favoured place,…, and 9 is not in its favoured place. Thus, the required number of ways is the number of derangements of 9 objects, namely,

$$d_9 = e^{-1} \times 9! \approx 0.3679 \times 9!$$

$$= 133504$$ ∎

Example 4 *In how many ways can we arrange the numbers $1, 2, 3, \ldots, 10$ so that 1 is not in the first place, 2 is not in the second place, and so on, and 10 is not in the 10^{th} place?*

▶ The required number of ways is

$$d_{10} \approx (10!)(e^{-1}) \approx 10! \times 0.3679$$

$$\approx 13,35,036.$$ ∎

Example 5 *From the set of all permutations of n distinct objects, one permutation is chosen at random. What is the probability that it is not a derangement?*

▶ The number of permutations of n distinct objects is $n!$. The number of derangements of these objects is d_n. Therefore, the probability that a permutation chosen is *not* a derangement is

$$p = 1 - \frac{d_n}{n!} = 1 - \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right\}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - + \cdots - \frac{(-1)^n}{n!}.$$ ∎

Example 6 *There are n pairs of children's gloves in a box. Each pair is of a different colour. Suppose the right gloves are distributed at random to n children, and thereafter the left gloves are also distributed to them at random. Find the probability that* (i) *no child gets a matching pair,* (ii) *every child gets a matching pair,* (iii) *exactly one child gets a matching pair, and* (iv) *at least 2 children get matching pairs.*

▶ Any one distribution of n right gloves to n children determines a set of n places for the n pairs of gloves. Let us take these as the natural places for the pairs of gloves. The left gloves can be distributed to n children in $n!$ ways.

(i) The event of no child getting a matching pair occurs if the distribution of the left gloves is a derangement. The number of derangements is d_n. Therefore, the required probability, in this case, is

$$p_1 = \frac{d_n}{n!} = \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

(ii) The event of every child getting a matching pair occurs in only one distribution of the left gloves. Therefore, the required probability, in this case, is $p_2 = \dfrac{1}{n!}$.

(iii) The event of exactly one child getting a matching pair occurs when only one left glove is in the natural place, and all others are in wrong places. The number of such distributions is d_{n-1}. The required probability, in this case, is

$$p_3 = \frac{d_{n-1}}{n!} = \frac{1}{n}\left\{1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right\}.$$

(iv) The event of at least 2 children getting a matching pair occurs if the event of no child or one child getting a matching pair *does not* occur. The probability, in this case, is $p_4 = 1 - (p_1 + p_3)$. ■

Example 10 *For the positive integers $1, 2, 3, \ldots n$, there are 11660 derangements where $1, 2, 3, 4, 5$ appear in the first five positions. What is the value of n?*

▶ The integers 1, 2, 3, 4 and 5 can be deranged in the first five places in d_5 ways; the last $n-5$ integers in d_{n-5} ways. Hence, the number of derangements (being considered) is $d_5 \times d_{n-5}$. This is given as 11660. Thus, we have $d_5 \times d_{n-5} = 11660$, so that

$$d_{n-5} = \frac{11660}{d_5} = \frac{11660}{44} = 265.$$

But $265 = d_6$. Thus, $n - 5 = 6$ so that $n = 11$. ∎

Example 11 *In how many ways can the integers $1, 2, 3, \ldots, 10$ be arranged in a line so that no even integer is in its natural place.*

▶ Let A_1 be the set of all permutations of the given integers where 2 is in its natural place, A_2 be the set of all permutations in which 4 is in its natural place, and so on. Then, the number of permutations where no even integer is in its natural place is $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$. This is given by *

$$|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_5}| = |S| - S_1 + S_2 - S_3 + S_4 - S_5 \tag{i}$$

We note that $|S| = 10!$.

Now, the permutations in A_1 are all of the form $b_1 b_3 b_4 \ldots b_{10}$, where $b_1\ b_3\ b_4 \ldots b_{10}$ is a permutation of 1,3,4,5... 10. As such, $|A_1| = 9!$.

Similarly,

$$|A_2| = |A_3| = |A_4| = |A_5| = 9!,$$

so that

$$S_1 = \Sigma|A_i| = 5 \times 9! = C(5, 1) \times 9!.$$

The permutations in $A_1 \cap A_2$ are all of the form $b_1 2 b_3 4 b_5 b_6 \ldots b_{10}$, where $b_1 b_3 b_5 b_6 \ldots b_{10}$ is a permutation of 1, 3, 5, 6, ... 10. As such, $|A_1 \cap A_2| = 8!$. Similarly, each of $|A_i \cap A_j| = 8!$, and there are $C(10, 2)$ such terms. Hence

$$S_2 = \Sigma|A_i \cap A_j| = C(5, 2) \times 8!.$$

Likewise, we find

$$S_3 = C(5, 3) \times 7!, \quad S_4 = C(5, 4) \times 6!, \quad S_5 = C(5, 5) \times 5!.$$

Accordingly, expression (i) gives the required number as

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_5}| &= 10! - C(5, 1) \times 9! + C(5, 2) \times 8! - C(5, 3) \times 7! \\ &\quad + C(5, 4) \times 6! - C(5, 5) \times 5! \\ &= 2170680.\end{aligned}$$

■

UNIT-5

COMBINATORICS

related to counting

★ Combinatorics: Is a branch of mathematics which deals with counting techniques.

★ Binomial theorem:-

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} x^{n-r} a^r$$

Let $n-r = n_1$; $r = n_2$

$$= \sum \frac{n!}{n_1!\,n_2!} x^{n_1} a^{n_2}$$

★ Multinomial theorem:-

$$(x_1 + x_2 + \dots + x_m)^n = \sum \frac{n!}{n_1!\,n_2!\dots n_m!} x_1^{n_1} x_2^{n_2} \dots x_m^{n_m}$$

NOTE: $n_1 + n_2 + \dots + n_m = n$

★ Problems:

1. Find the co-efficient of x^9y^3 in the expansion of $(2x-3y)^{12}$.

ans $(2x-3y)^{12} = \sum_{r=0}^{12} {}^{12}C_r\,(2x)^{12-r}(-3y)^r$

$$= \sum_{r=0}^{12} {}^{12}C_r\, 2^{12-r} x^{12-r} (-3)^r y^r$$

Coefficient of x^9y^3 : $r=3$

$$= {}^{12}C_3\, 2^9 (-3)^3$$

$$= \frac{12!}{9!\,3!} 2^9 (-27)$$

$$= \frac{12^2 \times 11 \times 10 \times 9!}{9! \times 3 \times 2} \times (-27)\, 2^9$$

$$= 2^{10}(110)(-27)$$

$$= -2970 \times 2^{10} = -3041280$$

2. Find the coefficient of x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$.

ans
$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} {}^{15}C_r \,(3x^2)^{15-r}\left(-\frac{2}{x}\right)^r$$

$$= \sum_{r=0}^{15} {}^{15}C_r \, 3^{15-r} (-2)^r \, x^{30-2r} \, x^{-r}$$

$$= \sum_{r=0}^{15} {}^{15}C_r \, 3^{15-r} (-2)^r \, x^{30-3r}$$

$r=10$

Coefficient of $x^0 = {}^{15}C_{10} \, 3^5 (-2)^{10}$

$$= \frac{15!}{10!\,5!} 3^5 (-2)^{10}$$

$$= \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10!}{10! \times 5 \times 4 \times 3 \times 2} \times 3^5 (-2)^{10}$$

$$= 747242496$$

3. Find the coefficient of x^{12} in the expansion of $x^3(1-2x)^{10}$.

ans
$$(1-2x)^{10} = \sum_{r=0}^{10} {}^{10}C_r \,(1)^{10-r} (-2x)^r$$

$$= \sum_{r=0}^{10} {}^{10}C_r \,(-2)^r x^r$$

$$x^3(1-2x)^{10} = \sum_{r=0}^{10} {}^{10}C_r \,(-2)^r x^{r+3}$$

$r=9$

Coefficient of x^{12} is: ${}^{10}C_9 (-2)^9 = \frac{10!}{1!\,9!}(-2)^9$

$$= 10 \times (-2)^9 = -5120$$

4. Find the coefficient of x^k in the expansion of $(1+x+x^2)(1+x)^n$ where $0 \le k \le n+2$.

ans $(1+x)^n = \sum_{r=0}^{n} {}^nC_r (1)^{n-r}(x)^r = \sum_{r=0}^{n} {}^nC_r x^r$

$(1+x+x^2)(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r(1+x+x^2)$

$= \sum_{r=0}^{n} {}^nC_r (x^r + x^{r+1} + x^{r+2})$

$= \sum {}^nC_r x^r + \sum {}^nC_r x^{r+1} + \sum {}^nC_r x^{r+2}$

$(r=k) \quad (r=k-1) \quad (r=k-2)$

Coefficient of $x^k = {}^nC_k + {}^nC_{k-1} + {}^nC_{k-2}$

$= \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!} + \frac{n!}{(n-k+2)!\,(k-2)!}$

5. Find the coefficient of $x^2y^3z^4w$ in the expansion of $(x-y-z+w)^{10}$.

ans $(x-y-z+w)^{10} = \sum \frac{10!}{n_1!n_2!n_3!n_4!} x^{n_1}(-y)^{n_2}(-z)^{n_3}(w)^{n_4}$

$= \sum \frac{10!}{n_1!n_2!n_3!n_4!} x^{n_1}(-1)^{n_2}(-1)^{n_3} y^{n_2} z^{n_3} w^{n_4}$

Coefficient of $x^2y^3z^4w$: $(n_1=2)$ $(n_2=3)$ $(n_3=4)$ $(n_4=1)$

$= \frac{10!}{2!3!4!1!}(-1)^3(-1)^4 = \frac{-10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2 \times 3 \times 2 \times 4!}$

$= -12600$

6. Find the coefficient of x^5 in the expansion of $(2-x+3x^2)^6$

ans $(2-x+3x^2)^6 = \sum \frac{6!}{n_1!n_2!n_3!} (2)^{n_1}(-x)^{n_2}(3x^2)^{n_3}$

$= \sum \frac{6!}{n_1!n_2!n_3!} (2)^{n_1}(-1)^{n_2}(x)^{n_2}(3)^{n_3} x^{2n_3}$

$= \sum \frac{6!}{n_1!n_2!n_3!} (2)^{n_1}(-1)^{n_2}(3)^{n_3}(x)^{n_2+2n_3}$

Coefficient of x^5: $n_2+2n_3=5$; $n_1+n_2+n_3=6$

Infinite values are possible.

7. Find the Coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

ans $(a+2b-3c+2d+5)^{16} = \sum \frac{16!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$

$= \sum \frac{16!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!}(2)^{n_2}(-3)^{n_3}\,2^{n_4}\,5^{n_5}\,a^{n_1}b^{n_2}c^{n_3}d^{n_4}$

Coefficient of $a^2b^3c^2d^5$: $n_1=2$, $n_2=3$, $n_3=2$, $n_4=5$, $n_5=4$

$= \frac{16!}{2!\,3!\,2!\,5!\,4!}\,2^3(-3)^2\,2^5\,5^4$

$= \frac{16\times15\times14\times13\times12\times11\times10\times9\times8\times7\times6\times5!}{4\times3\times2\times5!\times4\times3\times2}\,2^8(9)\,5^4 = 4.35891\times10^{14}$

8. Find coefficient of $x^2y^2z^3$ in the expansion of $(3x-2y-4z)^7$

ans $(3x-2y-4z)^7 = \sum \frac{7!}{n_1!\,n_2!\,n_3!}(3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$

$= \sum \frac{7!}{n_1!\,n_2!\,n_3!}\,3^{n_1}(-2)^{n_2}(-4)^{n_3}x^{n_1}y^{n_2}z^{n_3}$

Coefficient of $x^2y^2z^3$: $n_1=2$, $n_2=2$, $n_3=3$

$= \frac{7!}{2!\,2!\,3!}(3)^2(-2)^2(-4)^3$

$= \frac{7\times6\times5\times4\times3!\times9\times(-4)^3}{3!}$

$= -483840$

★ Catalan numbers :- defnⁿ is ※.

Consider the sequence b_0, b_1, b_2 ....etc. of positive integers defined by $b_0 = 1$ & $b_n = \frac{^{2n}C_n}{n+1} = \frac{C(2n,n)}{n+1}$

for $n \geq 1$. This sequence is called Catalan sequence & the terms of the sequence are called catalan numbers.

where $b_0 = 1$, $b_1 = 1$, $b_2 = 2$, $b_3 = 5$, $b_4 = 14$, $b_5 = 42$ .....

Consider x, y plane with O as origin & P as any point (n, n) where n is positive integer. In this plane, suppose we wish to reach point P starting from origin O & making two kinds of moves $R: (x,y) \to (x+1, y)$ & $U: (x,y) \to (x, y+1)$ also with a restriction that we may touch the line $y = x$ but cannot cross it. A path in which we move from O to P under the stated restriction is called good path. No. of such good paths is the catalan number, b_n.

★ Problem:

1. Using moves, $R: (x,y) \to (x+1, y)$ & $U: (x,y) \to (x, y+1)$, find no. of ways one can go from .
   i) $(0,0)$ to $(6,6)$ & not rise above the line $y = x$.
   ii) $(0,0)$ to $(9,9)$ & not rise above the line $y = x$.
   iii) $(2,1)$ to $(7,6)$ & not rise above the line $y = x - 1$
   iv) $(3,8)$ to $(10,15)$ & not rise above the line $y = x + 5$.

Ans: (i) No. of good paths from $(0,0)$ to $(6,6)$ is b_6.

$$b_6 = \frac{^{12}C_6}{7} = \frac{12!}{6!\,6!\,7} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6!}{6 \times 5 \times 4 \times 3 \times 2 \times 6! \times 7}$$

$$= \underline{132}$$

(ii) No. of good paths from $(0,0)$ to $(9,9)$ is b_9.

$$b_9 = \frac{^{18}C_9}{10} = \frac{18!}{9!\,9!\,10} = \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9!}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 9! \times 10}$$

$= 4862$

(iii) Shift origin (0,0) to (2,1) by using the transformation $X = x-2$ & $Y = y-1$. Then the point (7, 6) gets shifted to (5, 5) & the line $y = x-1$ shifts to $Y = X$.

$Y+1 = X+2-1$
$Y+1 = X+1$
$Y = X$

∴ No. of good paths from (2,1) to (7,6) & not rise above $y = x-1$ is = No. of good paths from (0,0) to (5,5) not rise above $Y=X$ is

$$b_5 = \frac{^{10}C_5}{6} = \frac{10!}{5!5!6} = \underline{42}$$

(iv) Shift origin (0,0) to (3,8) by using the transformation $X = x-3$ & $Y = y-8$. Then the point (10, 15) gets shifted to (7, 7) & the line $y = x+5$ shifts to $Y = X$

$Y+8 = X+3+5$
$Y = X$

∴ No. of good paths from (3,8) to (10,15) & not rise above $y = x+5$ = No. of good paths from (0,0) to (7,7) not rise above $Y=X$ is

$$b_7 = \frac{^{14}C_7}{8} = \frac{14!}{7!7!\times 8} = \underline{429}$$

☆ **Derangement:**
Permutation of n objects in which none of the objects are in its natural place is called derangement.

Eg:- Permutation of 1, 2, 3, ... etc in which 1 is not in the first place, 2 is not in the second place, ... n is not in n^{th} place is called derangement.

The no. of possible derangements of n distinct objects 1, 2, 3, ..., n is denoted by d_n. → If there is only one object, it continues to be in its natural place in every arrangement. ∴ $d_1 = 0$

→ If there are two objects, no. of derangements is 1.

→ For 3 objects ; 1, 2, 3, ... etc, no. of derangements is 2.

$$\therefore d_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right]$$

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

NOTE: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots = 0.3679$$

$$\therefore d_n = e^{-1} n! \quad \text{(when } n \text{ is very large)}$$

$$d_n = 0.3679 \times n! \quad \text{(don't use in exam)}$$

☆ Problems :-

1. Find the no. of derangements of 1, 2, 3, 4

ans: $d_4 = 4! \sum_{k=0}^{4} \frac{(-1)^4}{4!}$

$$= 4!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$= 4!\left[\frac{12 - 4 + 1}{4!}\right]$$

$$= 9$$

$(d_4 = 0.3679(4!) = 8.8 \approx 9)$

3. $d_{10} = 10! \sum_{k=0}^{10} \frac{(-1)^{10}}{10!}$

$$= 10!\left[1 - 1 + \frac{1}{2!} + \frac{(-1)}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\right]$$

$$= 1334961$$

$(d_{10} = 0.3679(10!) = 1335035.52)$

4. For the positive integers 1, 2, 3, ..., n there are 11660 derangements where 1, 2, 3, 4, 5 appear in the 1st five positions. What is the value of 'n'?

ans $\rightarrow d_5 \times d_{n-5} = 11660$ —— ①

$d_5 = 5!\left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$

$= 5!\left[\frac{11}{30}\right] = \underline{\underline{44}}$

From ①, $d_{n-5} = \frac{11660}{44} = 265$

$n - 5 = 6$

$\underline{\underline{n = 11}}$

$d_n = 0.3679\, n!$

$d_6 = 0.3679 \times 6!$

$= 265$

Given 1 to 5 should be in first five places. ∴ No. of derangements of 1, 2, 3, 4, 5 is 44. Remaining n−5 numbers can be arranged in d_{n-5} ways.

∴ $d_5 \times d_{n-5} = 11660$

6. Total no. of events $= n!$

i) $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

$P = \frac{\text{No. of favourable events}}{\text{Total no. of events}} = \frac{d_n}{n!} = \underline{\underline{\sum_{k=0}^{n} \frac{(-1)^k}{k!}}}$

ii) No. of favourable events = 1

∴ $P = \underline{\underline{\frac{1}{n!}}}$

iii) No. of favourable events $= {}^{n}C_1\, d_{n-1}$ (any one child)

$P = \frac{{}^{n}C_1 \times d_{n-1}}{n!} = \frac{n\,(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}}{n!}$

(iv) No child gets matching pair $= d_n$

one child gets matching pair $= d_{n-1} \times n$

$$P(\geq 2) = 1 - P(<2)$$

$$= 1 - [P(0) + P(1)]$$

$$= 1 - \left[\frac{d_n}{n!} + \frac{n d_{n-1}}{n!}\right]$$

$$= 1 + \left[\sum_{k=0}^{n} \frac{(-1)^k}{k!} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\right]$$

7. Total no. of ~~students~~ (ways) $(n) = 30$.

i) No one is grading his own paper.

Favorable events $= d_{30} = 30! \sum_{k=0}^{30} \frac{(-1)^k}{k!}$

$$P = \frac{30! \sum_{k=0}^{30} \frac{(-1)^k}{k!}}{30!}$$

ii) No. of favourable events $= 1$

$$P = \frac{1}{30!}$$

iii) Favourable events $= 30_{C_1} \times d_{29}$

$$P = \frac{30 \times d_{29}}{30!}$$

8. ~~Total no. of ways~~ (i). $d_{10} = 0.3679 \times 10! = 1335035.52$

$\simeq 1335036$

(ii) Atleast one man gets his umbrella

$= 10c_1 d_9 + 10c_2 d_8 + 10c_3 d_7 + \dots + 1$

$= 10! - d_{10}$

(iii) Atleast two of them gets their own umbrella:

$$= 10! - d_{10} - 10 d_9$$

$$\left\{ \geq 2 = 10! - [(=1) + (=0)] \right.$$

$$\left. = 10! - [10c_1 \times d_9 + d_{10}] \right\}$$

16/1/25

★ Principle of Inclusion & Exclusion:-

If S is a finite set & if A & B are subsets of S then

$|A \cup B| = |A| + |B| - |A \cap B|$

Also $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$|\overline{A \cup B}| = |\bar{A}| \cap |\bar{B}|$

$$\left.\begin{aligned} |\overline{A \cup B}| &= |S| - |A \cup B| \\ |\bar{A} \cap \bar{B}| &= |S| - |A \cup B| \end{aligned}\right\} \text{—} (1)$$

(1) is called Principle of Inclusion Exclusion

★ Principle of Inclusion & Exclusion for n sets:

Let S be finite set & $A_1, A_2, \ldots, A_n$ be n subsets of S.

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$
$$- [|A_1 \cap A_2| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|]$$
$$+ [|A_1 \cap A_2 \cap A_3| + \cdots]$$
$$\vdots$$
$$+ (-1)^{k-1} [|A_1 \cap A_2 \cap \cdots \cap A_n|]$$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n| = |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

★ Problems:-

II 1. ans: $S = 1200 +$

A → No. of students who took economics

B → —— 11 —— English

C → —— 11 —— Math

$|A| = 582$, $|A \cap B| = 217$

$|B| = 627$, $|A \cap C| = 307$

$|C| = 543$, $|B \cap C| = 250$

$|A \cap B \cap C| = 212$

$|\bar{A} \cap \bar{B} \cap \bar{C}| = ?$

$|A \cup B \cup C| = |A| + |B| + |C| - \{|A \cap B| + |B \cap C| + |C \cap A|\} + |A \cap B \cap C|$

$= 582 + 627 + 543 - \{217 + 307 + 250\} + 212$

$= 1190$

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - |A \cup B \cup C| = 1200 - 1190 = \underline{10}$

5.ans $|A| = \frac{300}{5}$ (Numbers b/w 1 & 300 divisible by 5)

$= 60$

$|B| = \frac{300}{6} = 50$

$|C| = \frac{300}{8} = 37.5 = 37$ (Don't round off; just take no. before decimal)

TAKE LCM

$|A \cap B| = \frac{300}{30} = 10$; $|B \cap C| = \frac{300}{24} = 12.5 = 12$; $|C \cap A| = \frac{300}{40} = 7.5 = 7$

$|A \cap B \cap C| = \left(\frac{300}{120}\right) = 2.5 = 2.$

a) $|A \cup B \cup C| = |A| + |B| + |C| - \{|A \cap B| + |B \cap C| + |C \cap A|\} + |A \cap B \cap C|$

$= (60 + 50 + 37) - (10 + 12 + 7) + 2 = \underline{\underline{120}}$

b) $|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - |A \cup B \cup C| = 300 - 120 = 180$

6.ans 5 a's, 4 b's, 3 c's

$|S| = \frac{12!}{5!\,4!\,3!}$

A, B, C are words in which a's, b's, c's are together respectively.

$|A| = \frac{8!}{4!\,3!}$ → All 5 a's → consider as (1 entity so $\underset{(5)}{1} + 4 + 3$)

$|B| = \frac{9!}{5!3!}$

$|C| = \frac{10!}{5!4!}$

→ 1 block of a's + 1 block of b's + 3 c's

$|A \cap B| = \frac{5!}{3!}$

$|B \cap C| = \frac{7!}{5!}$

$|C \cap A| = \frac{6!}{4!}$

$|A \cap B \cap C| = 3!$

$|A \cup B \cup C| = |A| + |B| + |C| - \{|A \cap B| + |B \cap C| + |C \cap A|\} + |A \cap B \cap C|$

$= \frac{8!}{4!3!} + \frac{9!}{5!3!} + \frac{10!}{5!4!} - \left\{\frac{5!}{3!} + \frac{7!}{5!} + \frac{6!}{4!}\right\} + 3!$

$=$

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - |A \cup B \cup C| = \frac{12!}{5!4!3!} -$

$=$

7. ans $|S| = 26!$

A, B, C, D → are set of words in which CAR, DOG, PUN, BYTE are together respectively

$|A| = 24!$ Take out the 3 letters (C, A, R) from 26 alphabets & add 1 as we consider CAR to be 1 block. (26−3+1)

$|B| = 24!$

$|C| = 24!$

$|D| = 23!$ (26−4+1)

$|A \cap B| = 22!$ (26−3−3+1+1)

$|B \cap C| = 22!$ (26−3−3+1+1)

$|A \cap C| = 22!$ $(26-3-3+1+1)$

$|A \cap D| = 21!$ $(26-3-4+1+1)$ $|x \cap y| = {}^4C_2 = 6$ total. (2 at a time intersecⁿ)

$|B \cap D| = 21!$ $(26-3-4+1+1)$

$|C \cap D| = 21!$ $(26-3-4+1+1)$

$|A \cap B \cap C| = 20!$ $(26-3-3-3+3)$

$|A \cap B \cap D| = 19!$ $(26-3-3-4+3)$

$|A \cap C \cap D| = 19!$ $(26-3-3-4+3)$

$|B \cap C \cap D| = 19!$ $(26-3-3-4+3)$

$|A \cap B \cap C \cap D| = 17!$ $(26-3-3-3-4+4)$

$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = |S| - |A \cup B \cup C \cup D|$

$|A|+|B|+|C|+|D| - \{|A \cap B|$

$= 26! - [(3 \times 24! + 23!) - (3 \times 22! + 3 \times 21!) + (20! + 3 \times 19!) - 17!]$